A NONCOMMUTATIVE PROBABILITY THEORY

BY

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ABSTRACT. A noncommutative probability theory is developed in which no boundedness, finiteness, or "tracial" conditions are imposed. The underlying structure of the theory is a "probability algebra" $(\mathcal{C}, \omega)$ where $\mathcal{C}$ is a *-algebra and ω is a faithful state on $\mathcal{C}$. Conditional expectations and coarse-graining are discussed. The bounded and unbounded commutants are considered and commutation theorems are proved. Two classes of probability algebras, which we call closable and symmetric probability algebras are shown to have important regularity properties. The canonical algebra of quantum mechanics is considered in some detail and a strong commutation theorem is proven for this case. Moreover, in this case, isotropic normal states, KMS states, and stable states are defined and characterized.

1. Introduction. This paper develops a framework for a noncommutative probability theory which encompasses both the classical and quantum theories. Unlike previous work in this direction, we do not impose boundedness, finiteness, or "tracial" conditions [5], [8], [12], [16], [18]. This enables us to treat unbounded observables which, although occurring naturally in quantum mechanics, have been largely avoided in previous theories of this type. The paper falls into two main parts. The first part (§§1–6) develops the concepts and results of the abstract theory. The second part (§§7–9) illustrates some of these concepts and extends results of the first part for a specific example which is of particular interest in quantum mechanics.

A *-algebra $(\mathcal{C}, *)$ is an associative, distributive, complex algebra $\mathcal{C}$ with an identity 1 and an involution * satisfying

$$x^{**} = x, \quad (xy)^* = y^*x^*, \quad \text{and} \quad (\alpha x + y)^* = \alpha^* x^* + y^*,$$

for all $x, y \in \mathcal{C}$, $\alpha \in \mathbf{C}$, where α^* denotes the complex conjugate of $\alpha \in \mathbf{C}$. A linear functional $\omega: \mathcal{C} \to \mathbf{C}$ which is positive ($\omega(x^*x) \geqslant 0$ for all $x \in \mathcal{C}$) and normalized ($\omega(1) = 1$) is called a *state*. A state ω is *faithful* if $\omega(x^*x) = 0$ implies $x = 0$. The triple $(\mathcal{C}, *, \omega)$ where ω is a faithful state is called a *probability algebra*. We now illustrate this concept with two examples.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\mathcal{C}$ be the set of complex-valued, measurable simple functions. We identify functions that are equal almost everywhere so $\mathcal{C}$ is actually composed of equivalence classes of simple functions. Under the pointwise operations of addition, multiplication, scalar multiplication, and complex conjugation $*$, $(\mathcal{C}, *)$ becomes a $*$-algebra. Define $\omega: \mathcal{C} \to \mathbf{C}$ by $\omega(\Sigma \, \alpha_i \chi_{A_i}) = \Sigma \, \alpha_i \mu(A_i)$. Then ω is a faithful state and $(\mathcal{C}, *, \omega)$ is a probability algebra. It is usually necessary to complete $\mathcal{C}$ to a larger, more useful, class of functions. This is conveniently done by defining the inner product $\langle x, y \rangle = \omega(y^*x)$ and forming the Hilbert space completion $H = L^2(\Omega, \mathcal{F}, \mu)$ of $\mathcal{C}$. Since ω is continuous on $\mathcal{C}$, ω can be extended uniquely to a continuous linear functional on H.

In quantum mechanics one frequently has a complex algebra $\mathcal{C}$ of operators on a common dense invariant domain D of a Hilbert space H which satisfies: $I|D \in \mathcal{C}$ where I is the identity operator; if $A^\dagger$ denotes the adjoint of A, then for all $A \in \mathcal{C}$, $D \subseteq D(A^\dagger)$ and $A^* = A^\dagger | D \in \mathcal{C}$. Then, under the usual operations, $(\mathcal{C}, *)$ is a $*$-algebra. Let $\psi \in D$ be a unit vector such that $A\psi = 0$ implies $A = 0$. If we define $\omega(A) = \langle A\psi, \psi \rangle$, then ω is a faithful state and $(\mathcal{C}, *, \omega)$ is a probability algebra.

2. Conditional expectation. Let $(\mathcal{C}, *, \omega)$ be a probability algebra and define the inner product $\langle x, y \rangle = \omega(y^*x)$. Notice that $\langle xy, z \rangle = \langle y, x^*z \rangle$ for all x, $y, z \in \mathcal{C}$. Let H be the Hilbert space completion of $\mathcal{C}$. Since ω is continuous on $\mathcal{C}$, it has a unique, continuous, linear extension to H which we also denote by ω. For $x \in \mathcal{C}$, define the operator $\pi(x): \mathcal{C} \to \mathcal{C}$ by $\pi(x)y = xy$. Then π is a $*$-representation of $\mathcal{C}$ on the dense invariant domain $D(\pi) = \mathcal{C} \subseteq H$ [13]. The *adjoint representation* π^* is defined as follows:

$$D(\pi^*) = \cap \left\{ D(\pi(x)^*): x \in \mathcal{C} \right\}, \quad \pi^*(x) = \left[\pi(x^*) \right]^* | D(\pi^*)$$

[13]. Notice that $\mathcal{C} = D(\pi) \subseteq D(\pi^*)$ and $\pi \subseteq \pi^*$.

Denote the set of operators $\{C: \mathcal{C} \to H\}$ by $L(\mathcal{C})$ and the set of bounded operators on H by $L(H)$. An operator $C \in L(\mathcal{C})$ is *symmetric* (*nonnegative*) if $\langle Cx, y \rangle = \langle x, Cy \rangle$ ($\langle Cx, x \rangle \geqslant 0$) for all $x, y \in \mathcal{C}$. Notice that a nonnegative operator is symmetric. Define the *unbounded commutant* $\pi(\mathcal{C})^c$ of π as

$$\pi(\mathcal{C})^c = \left\{ C \in L(\mathcal{C}): \langle C\pi(x)y, z \rangle = \langle Cy, \pi(x^*)z \rangle, x, y, z \in \mathcal{C} \right\}.$$

The commutant $\pi(\mathcal{C})'$ of π is

$$\pi(\mathcal{C})' = \left\{ C \in L(H): C|\mathcal{C} \in \pi(\mathcal{C})^c \right\}.$$

If $C \in \pi(A)^c$, then $C: \mathcal{C} \to D(\pi^*)$ and $C\pi(x) = \pi^*(x)C$ for all $x \in \mathcal{C}$ [13].

A linear functional $F: \mathcal{C} \to \mathbf{C}$ is *absolutely continuous* (denoted $F < \omega$) if $\omega(x_i^*x_i) \to 0$ implies $F(yx_i) \to 0$ for every $y \in \mathcal{C}$. If ν is a state on $(\mathcal{C}, *, \omega)$ we say that ν is *dominated* (denoted $\nu \prec \omega$) if there exists an $M > 0$ such that $\nu(x^*x) \leqslant M\omega(x^*x)$ for all $x \in \mathcal{C}$. Applying Schwarz's inequality,

$$|\nu(yx_i)| \leqslant \nu(yy^*)^{1/2}\nu(x_j^*x_j)^{1/2},$$

we see that $\nu \prec \omega$ implies $\nu < \omega$.

To illustrate these concepts, let $(\mathcal{C}, *, \omega)$ be the probability algebra of simple functions on $(\Omega, \mathcal{F}, \mu)$ considered in §1. If ν is a state on $(\mathcal{C}, *, \omega)$, then ν determines a measure $\nu_0(A) = \nu(\chi_A)$, $A \in \mathcal{F}$. By the way $(\mathcal{C}, *, \omega)$ is defined, ν_0 is automatically absolutely continuous relative to μ in the sense of measures. Now suppose $\nu < \omega$. Since ν is continuous on H, by the Riesz theorem, there exists an $f \in H$ such that $\nu(x) = \langle x, f \rangle = \int fx\, d\mu$. Hence $\nu_0(A) = \int_A f\, d\mu$ and $d\nu_0/d\mu = f \in H$. We thus see that $\nu < \omega$ if and only if ν_0 is absolutely continuous relative to μ and $d\nu_0/d\mu \in H$. Suppose $\nu \prec \omega$. If $\mathcal{C} \ni x_i \geqslant 0$ and $x_i \to 0$ in $L^1(\Omega, \mathcal{F}, \mu)$, then

$$\nu(x_i) = \nu(x_i^{1/2}x_i^{1/2}) \leqslant M\omega(x_i^{1/2}x_i^{1/2}) = M\int |x_i|\, d\mu \to 0.$$

In general, if $\mathcal{C} \ni x_i \to 0$ in $L^1(\Omega, \mathcal{F}, \mu)$, then x_i is a linear combination of four nonnegative functions all of which approach 0 in $L^1(\Omega, \mathcal{F}, \mu)$ so by linearity, $\nu(x_i) \to 0$. Hence ν can be extended to a linear functional in the dual $L^\infty(\Omega, \mathcal{F}, \mu)$ of $L^1(\Omega, \mathcal{F}, \mu)$. Thus there is an essentially bounded function f such that $\nu(x) = \int fx\, d\mu$. We then see that $\nu \prec \omega$ if and only if ν_0 is absolutely continuous relative to μ and $d\nu_0/d\mu$ is essentially bounded.

THEOREM 1. (a) *A linear functional F on $(\mathcal{C}, *, \omega)$ is absolutely continuous if and only if there exists a $C \in \pi(\mathcal{C})^c$ such that $F(x) = [\omega(Cx^*)]^*$ for all $x \in \mathcal{C}$, C is unique and is denoted $C = dF/d\omega$.*

(2) *A functional ν on $(\mathcal{C}, *, \omega)$ is an absolutely continuous state if and only if there exists a nonnegative $C \in \pi(\mathcal{C})^c$ such that $\omega(C1) = 1$ and $\nu(x) = \omega(Cx)$ for all $x \in \mathcal{C}$, C is strictly positive if and only if ν is faithful. The following statements are equivalent: (a) C is bounded; (b) $\nu \prec \omega$; (c) $\omega(x_i^*x_i) \to 0$ implies $\nu(x_i^*x_i) \to 0$.*

PROOF. (1) Suppose $F < \omega$. Then for any $y \in \mathcal{C}$, $x \to F(y^*x)$ is a continuous linear functional on $\mathcal{C}$. By the Riesz theorem, there exists a $y' \in H$ such that $F(y^*x) = \langle x, y' \rangle$ for every $x \in \mathcal{C}$. Define $C: \mathcal{C} \to H$ by $Cy = y'$. It is straightforward to show that C is linear. To show $C \in \pi(\mathcal{C})^c$, for every $x, y, z \in \mathcal{C}$ we have

$$\langle C\pi(x)y, z \rangle = \langle z, Cxy \rangle^* = [F(y^*x^*z)]^*$$
$$= \langle x^*z, Cy \rangle^* = \langle Cy, \pi(x^*)z \rangle.$$

Furthermore,

$$F(x) = \langle x, C1 \rangle = \langle \pi(x)1, C1 \rangle = \langle 1, Cx^* \rangle$$
$$= \langle Cx^*, 1 \rangle^* = [\omega(Cx^*)]^*.$$

Conversely, suppose $F(x) = \omega(Cx^*)^*$ for $C \in \pi(\mathcal{C})^c$. Then

$$|F(yx)| = |\omega(Cx^*y^*)| = |\langle Cx^*y^*, 1 \rangle| = |\langle Cy^*, x \rangle| \leqslant \|Cy^*\| \, \|x\|$$

so $F < \omega$. To show C is unique, suppose $F(x) = \omega(C_1 x^*)^*$ for $C_1 \in \pi(\mathcal{Q})^c$. Then for $x, y \in \mathcal{Q}$ we have

$$\langle C_1 y, x \rangle = \langle C_1 x^* y, 1 \rangle = \omega(C_1 x^* y) = [F(y^* x)]^*$$
$$= \omega(Cx^* y) = \langle Cy, x \rangle.$$

(2) Suppose ν is a state and $\nu < \omega$. Then by (1) there exists a $C \in \pi(\mathcal{Q})^c$ such that $\nu(y^* x) = \langle x, Cy \rangle$ for every $x, y \in \mathcal{Q}$. Now C is nonnegative since $\langle x, Cx \rangle = \nu(x^* x) \geqslant 0$ and $\omega(C1) = \langle C1, 1 \rangle = \nu(1) = 1$. Also, $\nu(x) = \langle x, C1 \rangle = \langle Cx, 1 \rangle = \omega(Cx)$. Conversely, if $\nu(x) = \omega(Cx)$ where $C \in \pi(\mathcal{Q})^c$ is nonnegative and $\omega(C1) = 1$, then ν is a state since

$$\nu(x^* x) = \omega(Cx^* x) = \langle Cx^* x, 1 \rangle = \langle Cx, x \rangle \geqslant 0.$$

That $\nu < \omega$ follows from (1). Since $\nu(x^* x) = \langle Cx, x \rangle$ we see that ν is faithful if and only if C is strictly positive. (a) $\Rightarrow$ (b) If C is bounded, then

$$\nu(x^* x) = \langle Cx, x \rangle \leqslant \|C\| \, \|x\|^2 = \|C\| \omega(x^* x).$$

(b) $\Rightarrow$ (c) trivial. (c) $\Rightarrow$ (a) Suppose $\omega(x_i^* x_i) \to 0$ implies $\nu(x_i^* x_i) \to 0$. Since C is nonnegative and densely defined, C has a selfadjoint extension and, hence, C has a nonnegative square root T. If $\mathcal{Q} \ni x_i \to 0$ then $\omega(x_i^* x_i) \to 0$ so

$$\langle Tx_i, Tx_i \rangle = \langle T^2 x_i, x_i \rangle = \langle Cx_i, x_i \rangle = \nu(x_i^* x_i) \to 0.$$

Hence $Tx_i \to 0$ and T is bounded on $\mathcal{Q}$. Therefore, $C = T^2$ is bounded. $\quad\square$

COROLLARY. $\nu \prec \omega$ if and only if there exists a nonnegative $C \in \pi(\mathcal{Q})'$ satisfying $\omega(C1) = 1$ and $\nu(x) = \omega(Cx)$ for all $x \in \mathcal{Q}$.

COROLLARY. If $\nu < \omega$ then there exists a unique $\hat{z} \in D(\pi^*)$ such that $\nu(x) = \omega[\pi^*(x)\hat{z}]$ for all $x \in \mathcal{Q}$.

PROOF. $\nu(x) = \omega(Cx) = \omega[C\pi(x)1] = \omega[\pi^*(x)C1]$. $\quad\square$

COROLLARY. If $\nu, \eta < \omega$ and $(d\nu/d\omega)1 = (d\eta/d\omega)1$ then $\nu = \eta$.

Let $(\mathcal{Q}, *, \omega)$ be a probability algebra and let $\mathcal{B} \subseteq \mathcal{Q}$ be a *-subalgebra. Then $\mathcal{B}$ is a subspace of H and its closure is a closed subspace $\overline{\mathcal{B}}$. Let $P_{\mathcal{B}}$ denote the orthogonal projection onto $\overline{\mathcal{B}}$. We denote the restriction of π to $\mathcal{B}$ with domain the dense subspace $\mathcal{B}$ of $\overline{\mathcal{B}}$ by $\pi|\mathcal{B}$. Thus $\pi|\mathcal{B}$ is a *-representation of $\mathcal{B}$ with dense invariant domain $D(\pi|\mathcal{B}) = \mathcal{B} \subseteq \overline{\mathcal{B}}$. For $x \in \mathcal{Q}$, the *conditional expectation* $E(x|\mathcal{B})$ of x given $\mathcal{B}$ is defined by

$$E(x|\mathcal{B}) \in D[(\pi|\mathcal{B})^*] \subseteq \overline{\mathcal{B}}, \tag{2.1}$$

$$\omega(yx) = \omega[(\pi|\mathcal{B})^*(y)E(x|\mathcal{B})] \quad \text{for all } y \in \mathcal{B}. \tag{2.2}$$

THEOREM 2. (1) $E(x|\mathcal{B})$ exists, is unique and $E(x|\mathcal{B}) = P_{\mathcal{B}} x$.
(2) A function $F: \mathcal{Q} \to D[(\pi|\mathcal{B})^*]$ equals $E(\cdot|\mathcal{B})$ if and only if F satisfies

$\omega[F(x)] = \omega(x)$ *and* $F(yx) = (\pi|\mathcal{B})^*(y)F(x)$ *for all* $x \in \mathcal{C}, y \in \mathcal{B}$.

(3) *If for all* $x \in \mathcal{C}$, *we define* $\omega_x: \mathcal{B} \to \mathbb{C}$ *by* $\omega_x(y) = \omega(xy)$, *then* $\omega_x < \omega|\mathcal{B}$ *and* $E(x|\mathcal{B}) = (d\omega_{x^*}/d\omega|\mathcal{B})1$.

PROOF. (1) To prove uniqueness, suppose E and E_1 are conditional expectations. Then for every $x \in \mathcal{C}, y \in \mathcal{B}$ we have

$$\langle E(x|\mathcal{B}), y \rangle = \langle E(x|\mathcal{B}), (\pi|\mathcal{B})(y)1 \rangle = \langle (\pi|\mathcal{B})(y)^* E(x|\mathcal{B}), 1 \rangle$$
$$= \langle (\pi|\mathcal{B})^*(y^*)E(x|\mathcal{B}), 1 \rangle = \omega[(\pi|\mathcal{B})^*(y^*)E(x|\mathcal{B})]$$
$$= \omega(y^*x) = \omega[(\pi|\mathcal{B})^*(y^*)E_1(x|\mathcal{B})] = \langle E_1(x|\mathcal{B}), y \rangle.$$

Hence, since $E(x|\mathcal{B})$, $E_1(x|\mathcal{B}) \in \overline{\mathcal{B}}$, $E(x|\mathcal{B}) = E_1(x|\mathcal{B})$ and E is unique. We now show that $P_{\mathcal{B}}$ is a conditional expectation. For every $x \in \mathcal{C}$, y, $z \in \mathcal{B}$ we have

$$|\langle \pi(z)y, P_{\mathcal{B}}x \rangle| = |\langle zy, x \rangle| = |\langle y, z^*x \rangle| < \|z^*x\| \, \|y\|.$$

Hence $P_{\mathcal{B}}x \in D(\pi(z)^*)$ for all z so $P_{\mathcal{B}}x \in D[(\pi|\mathcal{B})^*]$. To show that $P_{\mathcal{B}}x$ satisfies (2.2), for all $y \in \mathcal{B}$ we have

$$\omega[(\pi|\mathcal{B})^*(y)P_{\mathcal{B}}x] = \langle P_{\mathcal{B}}x, y^* \rangle = \langle x, y^* \rangle = \omega(yx).$$

(2) Letting $y = 1$ in (2.2) gives $\omega[E(x|\mathcal{B})] = \omega(x)$. Also, for all $x \in \mathcal{C}$, y, $z \in \mathcal{B}$ we have

$$\langle E(yx|\mathcal{B}), z \rangle = \omega[(\pi|\mathcal{B})^*(z)E(yx|\mathcal{B})] = \omega(zyx)$$
$$= \omega[(\pi|\mathcal{B})^*(zy)E(x|\mathcal{B})] = \omega[(\pi|\mathcal{B})^*(z)(\pi|\mathcal{B})^*(y)E(x|\mathcal{B})]$$
$$= \langle (\pi|\mathcal{B})^*(y)E(x|\mathcal{B}), z \rangle.$$

Hence $E(yx|\mathcal{B}) = (\pi|\mathcal{B})^*(y)E(x|\mathcal{B})$. Conversely, if F satisfies the requirements of (2) then for all $x \in \mathcal{C}, y \in \mathcal{B}$ we have

$$\omega[(\pi|\mathcal{B})^*(y)F(x)] = \omega[F(yx)] = \omega(yx).$$

By uniqueness, $F = E(\cdot|\mathcal{B})$.

(3) To show $\omega_x < \omega|\mathcal{B}$, suppose $x_i \in \mathcal{B}$ and $\omega(x_i^*x_i) \to 0$. If $y \in \mathcal{B}$ we have

$$|\omega_x(yx_i)| = |\omega(xyx_i)| < \omega(xyy^*x)^{1/2}\omega(x_i^*x_i)^{1/2} \to 0.$$

Hence

$$d\omega_{x^*}/d\omega|\mathcal{B} \in (\pi|\mathcal{B})(\mathcal{B})^c \quad \text{and} \quad (d\omega_{x^*}/d\omega|\mathcal{B})1 \in D[(\pi|\mathcal{B})^*].$$

Furthermore, for all $y \in \mathcal{B}$ we have

$$\omega[(\pi|\mathcal{B})^*(y)(d\omega_{x^*}/d\omega|\mathcal{B})1] = \omega[(d\omega_{x^*}/d\omega|\mathcal{B})(\pi|\mathcal{B})(y)1]$$
$$= \omega[(d\omega_{x^*}/dx|\mathcal{B})y] = [\omega_{x^*}(y^*)]^* = [\omega(x^*y^*)]^* = \omega(yx).$$

The result now follows from uniqueness. $\square$

3. Coarse-graining. Let the *-algebra $(\mathcal{C}, *)$ represent some physical or statistical system. Suppose we have an a priori distribution given by a faithful state ω. Then $(\mathcal{C}, *, \omega)$ is a probability algebra. We can now predict the behaviour of the system in terms of ω, but suppose we want to make a more precise prediction. In practice, it is impossible to measure all the observables in $\mathcal{C}$, so we make a partial measurement. That is, we measure observables in a *-subalgebra $\mathcal{B} \subseteq \mathcal{C}$. This gives a state ν on $\mathcal{B}$. We would now like to extend ν to a state ν_c on $\mathcal{C}$ which gives no additional information. If this is possible, we call ν_c a coarse-graining of ν. This procedure is frequently used in quantum mechanics and in statistical inference [8], [10].

More precisely, let $(\mathcal{C}, *, \omega)$ be a probability algebra and let $\mathcal{B} \subseteq \mathcal{C}$ be a *-subalgebra. Let ν be a state on $\mathcal{B}$ such that $\nu < \omega|\mathcal{B}$. A $(\mathcal{B}, \omega)$ coarse-graining of ν is a state ν_c on $\mathcal{C}$ satisfying:

(3.1) $\nu_c|\mathcal{B} = \nu$,

(3.2) $\nu_c < \omega$,

(3.3) $\nu_c(x) = \nu_c[E(x|\mathcal{B})]$ for all $x \in \mathcal{C}$.

A *-subalgebra $\mathcal{B} \subseteq \mathcal{C}$ is *positivity preserving* if for any $x \in \mathcal{C}$ there exists a sequence $y_i \in \mathcal{B}$ such that $y_i^* y_i \to P_{\mathcal{B}} x^* x$. In the case of the probability algebra $(\mathcal{C}, *, \omega)$ of measurable simple funcitons on $(\Omega, \mathcal{F}, \mu)$, any *-subalgebra $\mathcal{B} \subseteq \mathcal{C}$ is positivity preserving.

THEOREM 3. *Let $\mathcal{B} \subseteq \mathcal{C}$ be a *-subalgebra and let ν be a state on $\mathcal{B}$ such that $\nu < \omega|\mathcal{B}$.*

(1) *The following statements are equivalent:*

(a) *$A (\mathcal{B}, \omega)$ coarse-graining of ν exists;*

(b) *ν has an absolutely continuous extension ν_1 to $\mathcal{C}$ such that $(d\nu_1/d\omega)1 \in \overline{\mathcal{B}}$;*

(c) *$d\nu/d\omega|\mathcal{B}$ has a nonnegative extension in $\pi(\mathcal{C})^c$.*

(2) *If a $(\mathcal{B}, \omega)$ coarse-graining ν_c of ν exists it is unique and satisfies*

(d) *$\nu_c(x) = \nu[E(x|\mathcal{B})]$ for all $x \in \mathcal{C}$;*

(e) *ν_c is the unique absolutely continuous extension of ν to $\mathcal{C}$ such that $(d\nu_c/d\omega)1 \in \overline{\mathcal{B}}$;*

(f) *ν_c is the unique absolutely continuous extension of ν to $\mathcal{C}$ such that $(d\nu_c/d\omega)|\mathcal{B} = d\nu/d\omega|\mathcal{B}$.*

(3) *If $\nu \prec \omega|\mathcal{B}$ and $\mathcal{B}$ is positivity preserving, then ν_c exists and $\nu_c \prec \omega$.*

PROOF. (1) (a) $\Rightarrow$ (b) Suppose a $(\mathcal{B}, \omega)$ coarse-graining ν_c of ν exists. Then ν_c is an absolutely continuous extension of ν to $\mathcal{C}$. If $x \in \mathcal{C}$ and $\mathcal{B} \ni y_i \to P_{\mathcal{B}} x$ we have

$$\langle x, (d\nu_c/d\omega)1 \rangle = \langle (d\nu_c/d\omega)x, 1 \rangle = \omega\big[(d\nu_c/d\omega)x\big]$$

$$= \nu_c(x) = \nu_c(P_{\mathfrak{B}}x) = \lim \nu_c(y_i) = \lim \omega\big[(d\nu_c/d\omega)y_i\big]$$

$$= \lim \langle (d\nu_c/d\omega)y_i, 1 \rangle = \lim \langle y_i, (d\nu_c/d\omega)1 \rangle$$

$$= \langle P_{\mathfrak{B}}x, (d\nu_c/d\omega)1 \rangle = \langle x, P_{\mathfrak{B}}(d\nu_c/d\omega)1 \rangle.$$

Hence $(d\nu_c/d\omega)1 = P_{\mathfrak{B}}(d\nu_c/d\omega)1 \in \overline{\mathfrak{B}}$.

(b) $\Rightarrow$ (c) Suppose ν has an absolutely continuous extension ν_1 to $\mathfrak{C}$ such that $(d\nu_1/d\omega)1 \in \overline{\mathfrak{B}}$. Then $d\nu_1/d\omega \in \pi(\mathfrak{C})^c$, and for any $y \in \mathfrak{B}$ we have

$$\langle y, (d\nu_1/d\omega)1 \rangle = \nu_1(y) = \nu(y) = \nu(y) = \langle y, (d\nu/d\omega|\mathfrak{B})1 \rangle.$$

Hence, $(d\nu_1/d\omega)1 = (d\nu/d\omega|\mathfrak{B})1$ and for any $y \in \mathfrak{B}$ we have

$$(d\nu/d\omega|\mathfrak{B})y = (d\nu/d\omega|\mathfrak{B})\pi(y)1 = \pi^*(y)(d\nu/d\omega|\mathfrak{B})1$$

$$= \pi^*(y)(d\nu_1/d\omega)1 = (d\nu_1/d\omega)y.$$

Thus $d\nu_1/d\omega$ is a nonnegative extension of $d\nu/d\omega|\mathfrak{B}$ in $\pi(\mathfrak{C})^c$.

(c) $\Rightarrow$ (a) Suppose $d\nu/d\omega|\mathfrak{B}$ has a nonnegative extension $C \in \pi(\mathfrak{C})^c$. We now show that $\nu_c(x) = \nu[E(x|\mathfrak{B})]$ is a $(\mathfrak{B}, \omega)$ coarse-graining of ν. Clearly, ν_c is linear on $\mathfrak{C}$, $\nu_c(1) = 1$, and ν_c satisfies (3.1) and (3.3). If $x, y \in \mathfrak{C}$ and $\mathfrak{B} \ni z_i \to P_{\mathfrak{B}}yx$ we have

$$\nu_c(yx) = \nu\big[E(yx|\mathfrak{B})\big] = \nu(P_{\mathfrak{B}}yx) = \lim \nu(z_i)$$

$$= \lim \omega\big[(d\nu/d\omega|\mathfrak{B})z_i\big] = \lim \langle (d\nu/d\omega|\mathfrak{B})z_i, 1 \rangle$$

$$= \lim \langle z_i, (d\nu/d\omega|\mathfrak{B})1 \rangle = \langle P_{\mathfrak{B}}yx, (d\nu/d\omega|\mathfrak{B})1 \rangle$$

$$= \langle yx, (d\nu/d\omega|\mathfrak{B})1 \rangle = \langle \pi(y)x, C1 \rangle = \langle x, Cy^* \rangle.$$

Hence $\nu_c(x^*x) = \langle x, Cx \rangle \geqslant 0$ so ν_c is a state and $|\nu_c(yx)| \leqslant \|Cy^*\| \, \|x\|$ so $\nu_c < \omega$ and (3.2) is satisfied.

(2) (d) Suppose a $(\mathfrak{B}, \omega)$ coarse-graining ν_c of ν exists. For $x \in \mathfrak{C}$, let $\mathfrak{B} \ni y_i \to E(x|\mathfrak{B})$. Then we have

$$\nu_c(x) = \nu_c\big[E(x|\mathfrak{B})\big] = \lim \nu_c(y_i) = \lim \nu(y_i) = \nu\big[E(x|\mathfrak{B})\big].$$

(e) That $(d\nu_c/d\omega)1 \in \overline{\mathfrak{B}}$ follows from (a) $\Rightarrow$ (b) above. If ν_1 is an absolutely continuous extension of ν to $\mathfrak{C}$ such that $(d\nu_1/d\omega)1 \in \overline{\mathfrak{B}}$, then from the proof of (b) $\Rightarrow$ (c) above we have $(d\nu_1/d\omega)1 = (d\nu_c/d\omega)1$. It follows from the third corollary of Theorem 1 that $\nu_1 = \nu_c$.

(f) This follows from the proof of (b) $\Rightarrow$ (c) and (e) above.

(3) Suppose $\nu \prec \omega|\mathfrak{B}$ and $\mathfrak{B}$ is positivity preserving. We will show that $\nu_c(x) = \nu[E(x|\mathfrak{B})]$ is a $(\mathfrak{B}, \omega)$ coarse-graining of ν and that $\nu_c \prec \omega$. To show that ν_c is positive, let $x \in \mathfrak{C}$ and let $y_i \in \mathfrak{B}$ be a sequence such that $y_i^*y_i \to P_{\mathfrak{B}}x^*x$. Then

$$\nu_c(x^*x) = \nu(P_{\mathfrak{B}}x^*x) = \lim \nu(y_i^*y_i) \geqslant 0.$$

To show that $\nu_c \prec \omega$, we have

$$\nu_c(x^*x) = \lim \nu(y_i^*y_i) = \lim \omega\big[(d\nu/d\omega|\mathcal{B})y_i^*y_i\big]$$

$$= \lim\langle(d\nu/d\omega|\mathcal{B})y_i, y_i\rangle \leqslant \|d\nu/d\omega|\mathcal{B}\|\lim\langle y_i, y_i\rangle$$

$$= \|d\nu/d\omega|\mathcal{B}\|\langle P_{\mathcal{B}}x^*x, 1\rangle = \|d\nu/d\omega|\mathcal{B}\|\omega(x^*x). \quad \square$$

We now show that in a certain sense the $(\mathcal{B}, \omega)$ coarse-graining of ν is the absolutely continuous extension of ν with minimal information. Let $(\mathcal{C}, *, \omega)$ be a probability algebra and suppose $\nu < \omega$ is a positive linear functional. We would like to define the information measure $I_\omega(\nu)$ of ν relative to ω. Guided by the properties of a measure we shall postulate that I_ω be nonnegative and additive. Nonnegativity is simply $I_\omega(\nu) \geqslant 0$. To motivate additivity, consider the classical situation in which ν_1, ν_2 are mutually singular measures on a probability space $(\Omega, \mathcal{F}, \mu)$ which are absolutely continuous relative to μ. Since ν_1, ν_2 are mutually singular there exist disjoint measurable sets A_1, A_2 such that $A_1 \cup A_2 = \Omega$ and $\nu_1(A_2) = \nu_2(A_1) = 0$. It now seems reasonable to assume that $I_\omega(\nu_1 + \nu_2) = I_\omega(\nu_1) + I_\omega(\nu_2)$. Notice that ν_1, ν_2 are mutually singular if and only if $\int(d\nu_1/d\mu)(d\nu_2/d\mu)\,d\mu = 0$.

This reasoning motivates the following definition of information. If $F < \omega$ is a linear functional, the information $I_\omega(F)$ of F relative to ω satisfies:

(3.4) $I_\omega(F) \geqslant 0$, $I_\omega(\omega) = 1$;

(3.5) for F_1, $F_2 < \omega$ with $\langle(dF_1/d\omega)1, (dF_2/d\omega)1\rangle = 0$, $I_\omega(F_1 + F_2) = I_\omega(F_1) + I_\omega(F_2)$.

THEOREM 4. (1) $I_\omega(F)$ exists, is unique and $I_\omega(F) = \|(dF/d\omega)1\|^2 = F[(dF/d\omega)1]$. (2) If the $(\mathcal{B}, \omega)$ coarse-graining of ν exists, it is the unique absolutely continuous extension of ν to $\mathcal{C}$ with minimal information relative to ω.

PROOF. (1) It is clear that $F \to \|(dF/d\omega)1\|^2$ satisfies (3.4) and (3.5). To prove uniqueness, suppose I_ω satisfies (3.4) and (3.5). For $y \in D(\pi^*)$ define the linear functional $F_y: \mathcal{C} \to \mathbf{C}$ by $F_y(x) = \langle x, y\rangle$. Now $F_y < \omega$ since for all $x, z \in \mathcal{C}$ we have

$$|F_y(zx)| = |\langle zx, y\rangle| = |\langle x, \pi(z)^*y\rangle| \leqslant \|\pi(z)^*y\|\,\|x\|.$$

Define the map $G: D(\pi^*) \to \mathbf{R}^+$ by $G(y) = I_\omega(F_y)$. Notice that $(dF_y/d\omega)1 = y$ since for all $x \in \mathcal{C}$ we have

$$\langle(dF_y/d\omega)1, x\rangle = \langle(dF_y/d\omega)x^*, 1\rangle = \omega\big[(dF_y/d\omega)x^*\big]$$

$$= [F_y(x)]^* = \langle x, y\rangle^* = \langle y, x\rangle.$$

Let $y, z \in D(\pi^*)$ satisfy $\langle y, z\rangle = 0$. Then $\langle(dF_y/d\omega)1, (dF_z/d\omega)1\rangle = 0$ so

$$G(y + z) = I_\omega(F_{y+z}) = I_\omega(F_y + F_z)$$

$$= I_\omega(F_y) + I_\omega(F_z)$$

$$= G(y) + G(z).$$

Applying a result of [9] there exists a $c > 0$ such that $G(y) = c\|y\|^2$.

If $F < \omega$, then $F = F_{(dF/d\omega)1}$, since for all $x \in \mathcal{C}$ we have

$$F(x) = \left[\omega((dF/d\omega)x^*)\right]^* = \langle (dF/d\omega)x^*, 1\rangle^*$$
$$= \langle (dF/d\omega)1, x\rangle^* = \langle x, (dF/d\omega)1\rangle.$$

Hence

$$I_\omega(F) = I_\omega\left[F_{(dF/d\omega)1}\right] = G\left[(dF/d\omega)1\right] = c\|(dF/d\omega)1\|^2.$$

Also, $1 = I_\omega(\omega) = c\|(d\omega/d\omega)1\|^2 = c$. Finally, if $F < \omega$ and $\mathcal{C} \ni x_i \to (dF/d\omega)1$ we have

$$I_\omega(F) = \langle (dF/d\omega)1, (dF/d\omega)1\rangle = \lim\langle (dF/d\omega)1, x_i\rangle$$
$$= \lim\langle (dF/d\omega)x_i^*, 1\rangle = \lim \omega\left[(dF/d\omega)x_i^*\right]$$
$$= \lim\left[F(x_i)\right]^* = F\left[(dF/d\omega)1\right]^* = F\left[(dF/d\omega)1\right].$$

(2) Suppose the $(\mathcal{B}, \omega)$ coarse-graining ν_c of ν exists. Suppose $F < \omega$ and $F|\mathcal{B} = \nu$. Since $\overline{\mathcal{B}}$ is a closed subspace of H, we have $(dF/d\omega)1 = x + y$ where $x \in \overline{\mathcal{B}}$ and $y \in \overline{\mathcal{B}}^\perp$. For $z \in \mathcal{B}$, since $d\nu/d\omega|\mathcal{B} = (d\nu_c/d\omega)|\mathcal{B}$ we have

$$\langle x, z\rangle = \langle x + y, z\rangle = \langle (dF/d\omega)1, z\rangle = \langle (dF/d\omega)z^*, 1\rangle$$
$$= \omega\left[(dF/d\omega)z^*\right] = \left[F(z)\right]^* = \nu(z^*) = \omega\left[(d\nu/d\omega|\mathcal{B})z^*\right]$$
$$= \omega\left[(d\nu_c/d\omega)z^*\right] = \langle (d\nu_c/d\omega)z^*, 1\rangle = \langle (d\nu_c/d\omega)1, z\rangle.$$

By Theorem 3(e), $(d\nu_c/d\omega)1 \in \mathcal{B}$ so $x = (d\nu_c/d\omega)1$. Hence

$$I_\omega(F) = \|(dF/d\omega)1\|^2 = \|x\|^2 + \|y\|^2 \geqslant \|x\|^2$$
$$= \|(d\nu_c/d\omega)1\|^2 = I_\omega(\nu_c).$$

Hence $I_\omega(\nu_c)$ is a minimum and F has minimal information relative to ω if and only if $\|y\| = 0$ in which case $(dF/d\omega)1 = (d\nu_c/d\omega)1$ and by the third corollary of Theorem 1, $F = \nu_c$. $\square$

Let $(\mathcal{C}_1, *)$ and $(\mathcal{C}_2, *)$ be *-algebras. Let $\mathcal{C}_1 \otimes \mathcal{C}_2$ be the algebraic tensor product of $\mathcal{C}_1$ and $\mathcal{C}_2$ with product defined by $(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 y_1 \otimes x_2 y_2$. If we define $(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*$, then $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$ is a *-algebra. We call $(\mathcal{C}_1, *)$, $(\mathcal{C}_2, *)$ the *component* systems of the *compound* system $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$. If $(\mathcal{C}_1, *)$, $(\mathcal{C}_2, *)$ represent the observables for two physical systems, then $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$ represents the observables for the two systems combined. Let ω be a faithful state on $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$ so that $(\mathcal{C}_1 \otimes \mathcal{C}_2, *, \omega)$ becomes a probability algebra. Define the states ω_1, ω_2 on $\mathcal{C}_1, \mathcal{C}_2$, respectively, by $\omega_1(x_1) = \omega(x_1 \otimes 1)$, $\omega_2(x_2) = \omega(1 \otimes x_2)$. Then ω_1, ω_2 are faithful states called the *reduced* states. Conversely, if ω_1, ω_2 are faithful states on $\mathcal{C}_1, \mathcal{C}_2$, respectively, then we define the faithful state $\omega_1 \otimes \omega_2$ on $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$ by $\omega_1 \otimes \omega_2(x_1 \otimes x_2) = \omega_1(x_1)\omega_2(x_2)$. If $\omega = \omega_1 \otimes \omega_2$ we say that $(\mathcal{C}_1, *)$, $(\mathcal{C}_2, *)$ are *uncorrelated* relative to ω. Physically, this means that when the two

component systems are combined they do not interact.

THEOREM 5. *The following statements are equivalent*: (a) $(\mathcal{Q}_1, *)$, $(\mathcal{Q}_2, *)$ *are uncorrelated relative to* ω; (b) $E(x_1 \otimes x_2|\mathcal{Q}_1 \otimes 1) = \omega_2(x_2)x_1 \otimes 1$ *for all* $x_1 \in \mathcal{Q}_1$, $x_2 \in \mathcal{Q}_2$; (c) $E(x_1 \otimes x_2|\mathcal{Q}_1 \otimes 1) = f(x_1, x_2)x_1 \otimes 1$ *for all* $x_1 \in \mathcal{Q}_1$, $x_2 \in \mathcal{Q}_2$; (d) *for any state* ν *on* $\mathcal{Q}_1 \otimes 1$ *with* $\nu < \omega|\mathcal{Q}_1 \otimes 1$, *the* $(\mathcal{Q}_1 \otimes 1, \omega)$ *coarse-graining of* ν_c *of* ν *exists and* $\nu_c = \nu \otimes \omega_2$ (*identity* $\mathcal{Q}_1 \otimes 1$ *and* $\mathcal{Q}_1$ *in the natural way*).

PROOF. (a) $\Rightarrow$ (b) If (a) holds then $\omega_2(x_2)x_1 \otimes 1 \in \mathcal{Q}_1 \otimes 1$ and for any $y_1 \in \mathcal{Q}_1$ we have

$$\omega\big[(y_1 \otimes 1)\omega_2(x_2)x_1 \otimes 1\big] = \omega_2(x_2)\omega(y_1x_1 \otimes 1)$$
$$= \omega_1(y_1x_1)\omega_2(x_2) = \omega_1 \otimes \omega_2(y_1x_1 \otimes x_2)$$
$$= \omega\big[(y_1 \otimes 1)(x_1 \otimes x_2)\big].$$

Hence $E(x_1 \otimes x_2|\mathcal{Q}_1 \otimes 1) = \omega_2(x_2)x_1 \otimes 1$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) Suppose (c) holds. Then for $x_1, y_1 \in \mathcal{Q}_1$, $x_2 \in \mathcal{Q}_2$ we have

$$\omega(y_1x_1 \otimes x_2) = \omega\big[(y_1 \otimes 1)(x_1 \otimes x_2)\big]$$
$$= \omega\big[(y_1 \otimes 1)E(x_1 \otimes x_2|\mathcal{Q}_1 \otimes 1)\big]$$
$$= f(x_1, x_2)\omega(y_1x_1 \otimes 1) = f(x_1, x_2)\omega_1(y_1x_1).$$

Letting $y_1 = 1$ gives $f(x_1, x_2) = \omega(x_1 \otimes x_2)/\omega_1(x_1)$ and letting $x_1 = 1$ gives

$$\omega(y_1 \otimes x_2) = \omega(1 \otimes x_2)\omega_1(y_1) = \omega_1(y_1)\omega_2(x_2).$$

Hence $\omega = \omega_1 \otimes \omega_2$ and $(\mathcal{Q}_1, *)$, $(\mathcal{Q}_2, *)$ are uncorrelated relative to ω. Let $T = d\nu/d\omega|\mathcal{Q}_1 \otimes 1$. Then $T \otimes I$ is a nonnegative extension of T to $\mathcal{Q}_1 \otimes \mathcal{Q}_2$. We now show that $T \otimes I \in \pi(\mathcal{Q}_1 \otimes \mathcal{Q}_2)^c$. Indeed,

$$\langle T \otimes I\pi(x_1 \otimes x_2)y_1 \otimes y_2, z_1 \otimes z_2\rangle$$
$$= \langle Tx_1y_1 \otimes x_2y_2, z_1 \otimes z_2\rangle$$
$$= \langle Tx_1y_1, z_1\rangle\langle x_2y_2, z_2\rangle = \langle Ty_1, x_1^*z_1\rangle\langle y_2, x_2^*z_2\rangle$$
$$= \langle Ty_1 \otimes y_2, x_1^*z_1 \otimes x_2^*z_2\rangle$$
$$= \langle T \otimes Iy_1 \otimes y_2, (x_1 \otimes x_2)^*(z_1 \otimes z_2)\rangle$$
$$= \langle T \otimes Iy_1 \otimes y_2, \pi\big[(x_2 \otimes x_2)^*\big]z_1 \otimes z_2\rangle.$$

It follows from Theorem 3(c) that ν_c exists. Now $\nu_c = \nu \otimes \omega_2$ since from (b) we have

$$\nu_c(x_1 \otimes x_2) = \nu\big[E(x_1 \otimes x_2|\mathcal{Q}_1 \otimes 1)\big] = \nu\big[\omega_2(x_2)x_1 \otimes 1\big]$$
$$= \omega_2(x_2)\nu(x_1) = \nu \otimes \omega_2(x_1 \otimes x_2).$$

(d) $\Rightarrow$ (a) If (d) holds then $\omega = \omega_{1_c} = \omega_1 \otimes \omega_2$. $\quad\square$

Theorem 5(d) has the following physical interpretation. Suppose $(\mathcal{C}_2, *)$ represents a physical system whose state we want to determine. We let $(\mathcal{C}_2, *)$ interact with a measuring device represented by $(\mathcal{C}_1, *)$ to get a compound system $(\mathcal{C}_1 \otimes \mathcal{C}_2, *)$. Let ω be an a priori state giving a probability algebra $(\mathcal{C}_1 \otimes \mathcal{C}_2, *, \omega)$. A measurement of $(\mathcal{C}_1, *)$ gives a state ν on $\mathcal{C}_1 \otimes 1$ such that $\nu < \omega | \mathcal{C}_1 \otimes 1$. The $(\mathcal{C}_1 \otimes 1, \omega)$ coarse-graining ν_c of ν gives the "most likely" state of the compound system. The reduced state ν_{c^2} gives the "most likely" state of the physical system $(\mathcal{C}_2, *)$. If $(\mathcal{C}_1, *)$, $(\mathcal{C}_2, *)$ are uncorrelated relative to ω, then we have seen that $\nu_c = \nu \otimes \omega_2$ so that $\nu_{c^2} = \omega_2$. Hence, in this case, the measurement gives no additional information about the state of the system.

4. The unbounded commutant. We have seen the importance of the unbounded commutant $\pi(\mathcal{C})^c$ in the previous sections. In this section we characterize $\pi(\mathcal{C})^c$.

Let $(\mathcal{C}, *, \omega)$ be a probability algebra. For $y \in D(\pi^*)$ define $\pi^c(y) \in L(\mathcal{C})$ by $\pi^c(y)x = \pi^*(x)y$ for every $x \in \mathcal{C}$. A sequence of operators $C_i \in L(\mathcal{C})$ *converges weakly* to $C \in L(\mathcal{C})$ if $\langle C_i x, y \rangle \to \langle Cx, y \rangle$ for every x, $y \in \mathcal{C}$.

THEOREM 6. *π^c is a weakly continuous linear bijection from $D(\pi^*)$ onto $\pi(\mathcal{C})^c$.*

PROOF. Clearly, π^c is linear. To show that π^c maps $D(\pi^*)$ into $\pi(\mathcal{C})^c$, for $y \in D(\pi^*)$, $x, z, u \in \mathcal{C}$ we have

$$\langle \pi^c(y)\pi(x)z, u \rangle = \langle \pi^c(y)xz, u \rangle = \langle \pi^*(xz)y, u \rangle$$
$$= \langle \pi^*(x)\pi^*(z)y, u \rangle = \langle \pi^*(z)y, \pi(x^*)u \rangle$$
$$= \langle \pi^c(y)z, \pi(x^*)u \rangle.$$

To show that π^c is surjective, let $C \in \pi(\mathcal{C})^c$. Then $C1 \in D(\pi^*)$, and for every $x \in \mathcal{C}$ we have

$$\pi^c(C1)x = \pi^*(x)C1 = C\pi(x)1 = Cx.$$

To show that π^c is injective, suppose $y, y_1 \in D(\pi^*)$ and $\pi^c(y) = \pi^c(y_1)$. Then

$$y = \pi^*(1)y = \pi^c(y)1 = \pi^c(y_1)1 = \pi^*(1)y_1 = y_1.$$

To show that π^c is weakly continuous, suppose $y_i, y \in D(\pi^*)$ and $y_i \to y$ in norm. Then for every $x, z \in \mathcal{C}$ we have

$$\lim \langle \pi^c(y_i)x, z \rangle = \lim \langle \pi^*(y)y_i, z \rangle = \lim \langle y_i, \pi(x^*)z \rangle$$
$$= \langle y, \pi(x^*)z \rangle = \langle \pi^*(x)y, z \rangle = \langle \pi^c(y)x, z \rangle. \quad \square$$

The above theorem gives a characterization of $\pi(\mathcal{C})^c$ in the sense that $\pi(\mathcal{C})^c = \pi^c[D(\pi^*)]$. We shall later give a similar characterization of $\pi(\mathcal{C})'$,

but first we shall analyze the structure of $\pi(\mathcal{Q})^c$. Let M be a dense subspace of a Hilbert space H and let $S \subseteq L(M)$. We say that S is *weakly closed* if for any net $C_\alpha \in S$ which converges weakly to an operator $C \in L(M)$ we have $C \in S$. We call S an *op-algebra* if for all $A, B \in S$, $A: M \to M$ and $AB \in S$, $(\alpha A + B) \in S$ for all $\alpha \in \mathbf{C}$. For $A \in S$, denote the adjoint of A by $A^\dagger$, and if $D(A^\dagger) \supseteq M$ define $A^* = A^\dagger | M$. We call an op-algebra S an *op*-algebra* if $I|M \in S$ and if for all $A \in S$ we have $D(A^\dagger) \supseteq M$ and $A^* \in S$. If S is an op*-algebra, it is easy to see that $*$ is an involution on S and hence S is a *-algebra. A *-representation π of a *-algebra $(\mathcal{Q}, *)$ is *selfadjoint* if $\pi = \pi^*$ (i.e. $D(\pi) = D(\pi^*)$) [13].

Now let $(\mathcal{Q}, *, \omega)$ be a probability algebra. It is easy to see that $\pi(\mathcal{Q})^c$ is a weakly closed linear manifold with identity, in $L(\mathcal{Q})$. In general, $\pi(\mathcal{Q})^c$ need not be an op-algebra, let alone an op*-algebra. Theorem 8, to follow, characterizes when these circumstances hold.

LEMMA 7. *Let $(\mathcal{Q}, *, \omega)$ be a probability algebra. If there exists a map $x \to x^b$ on $\mathcal{Q}$ such that $\omega(y^*x) = \omega(xy^b)$ for all $x, y \in \mathcal{Q}$ then ^{b} is unique, is an involution, $\langle x^*, y \rangle = \langle y^b, x \rangle$ and $\langle xy, z \rangle = \langle x, zy^b \rangle$ for all $x, y, z \in \mathcal{Q}$.*

PROOF. To show ^{b} is unique, suppose $'$ is another map such that $\omega(y^*x) = \omega(xy')$ for all $x, y \in \mathcal{Q}$. Then for all $x, y \in \mathcal{Q}$ we have

$$\langle y', x \rangle = \omega(x^*y') = \omega(x^*y^b) = \langle y^b, x \rangle,$$

so $y' = y^b$. That ^{b} is an involution follows from the equations:

$$\omega\big[z(x + y)^b\big] = \omega\big[(x + y)^*z\big] = \omega(x^*z) + \omega(y^*z)$$
$$= \omega(zx^b) + \omega(zy^b) = \omega\big[z(x^b + y^b)\big];$$
$$\omega\big[z(xy)^b\big] = \omega\big[(xy)^*z\big] = \omega(y^*x^*z) = \omega(x^*zy^b) = \omega(zy^bx^b);$$
$$\omega\big[z(\alpha x)^b\big] = \omega\big[(\alpha x)^*z\big] = \omega(\alpha^*x^*z) = \omega(z\alpha^*x^b);$$
$$\omega(zx^{bb}) = \omega(x^{b*}z) = \omega\big[(z^*x^b)^*\big] = \big[\omega(z^*x^b)\big]^*$$
$$= \big[\omega(x^*z^*)\big]^* = \omega(zx).$$

Finally, we have

$$\langle x^*, y \rangle = \omega(y^*x^*) = \omega(x^*y^b) = \langle y^b, x \rangle$$

and

$$\langle xy, z \rangle = \omega(z^*xy) = \omega(xyz^b) = \omega\big[x(zy^b)^b\big]$$
$$= \omega\big[(zy^b)^*x\big] = \langle x, zy^b \rangle. \quad \square$$

Suppose there exists an involution ^{b} on the probability algebra $(\mathcal{Q}, *, \omega)$ such that $\omega(y^*x) = \omega(xy^b)$ for all $x, y \in \mathcal{Q}$. For $x \in \mathcal{Q}$ define the operator

$\rho(x) \in L(\mathcal{C})$ by $\rho(x)y = yx$. It is easy to check that $\rho: \mathcal{C} \to L(\mathcal{C})$ is a ^{b}-antirepresentation of $\mathcal{C}$.

THEOREM 8. *Let* $(\mathcal{C}, *, \omega)$ *be a probability algebra.*

(1) *The following statements are equivalent*:

(a) $\pi(\mathcal{C})^c$ *is an algebra*;

(b) $C1 \in \mathcal{C}$ *for every* $C \in \pi(\mathcal{C})^c$;

(c) *for any* $F < \omega$ *there exists a* $y \in \mathcal{C}$ *such that* $F(x) = \langle x, y \rangle$;

(d) π *is selfadjoint.*

(2) $\pi(\mathcal{C})^c$ *is a* *-algebra if and only if* π *is selfadjoint and there exists an involution* ^{b} *on* $\mathcal{C}$ *such that* $\omega(y^*x) = \omega(xy^b)$ *for all* $x, y \in \mathcal{C}$.

(3) *If* $\pi(\mathcal{C})^c$ *is a* *-algebra, then* ρ *is a weakly continuous* ^{b}-*anti-isomorphism of* $\mathcal{C}$ *onto* $\pi(\mathcal{C})^c$.

PROOF. (1) (a) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (c) Suppose (b) holds and $F < \omega$. By Theorem 1, there exists a $C \in \pi(\mathcal{C})^c$ such that $F(x) = [\omega(Cx^*)]^*$ for every $x \in \mathcal{C}$. If $y = C1 \in \mathcal{C}$, then for all $x \in \mathcal{C}$ we have

$$F(x) = [\omega(Cx^*)]^* = \langle Cx^*, 1 \rangle^* = \langle 1, Cx^* \rangle = \langle x, y \rangle.$$

(c) $\Rightarrow$ (d) Suppose (c) holds and $z \in D(\pi^*)$. As in the proof of Theorem 4, the linear functional $F(x) = \langle x, z \rangle$ is absolutely continuous. Hence, there exists a $z' \in \mathcal{C}$ such that $\langle x, z \rangle = \langle x, z' \rangle$ for all $x \in \mathcal{C}$. Thus $z = z' \in \mathcal{C}$, so $D(\pi^*) = \mathcal{C} = D(\pi)$ and π is selfadjoint.

(d) $\Rightarrow$ (a) Suppose (d) holds. Then $C: \mathcal{C} \to \mathcal{C}$ for every $C \in \pi(\mathcal{C})^c$. Furthermore, if $A, B \in \pi(\mathcal{C})^c$ then for all $x \in \mathcal{C}$ we have

$$AB\pi(x) = A\pi(x)B = \pi(x)AB$$

so $\pi(\mathcal{C})^c$ is an algebra.

(2) Suppose $\pi(\mathcal{C})^c$ is a *-algebra. By (1), π is selfadjoint. If $C \in \pi(\mathcal{C})^c$, then $C^* = C^\dagger | \mathcal{C} \in \pi(\mathcal{C})^c$ so $C^*: \mathcal{C} \to \mathcal{C}$. For $y \in \mathcal{C}$, by Theorem 6, $\pi^c(y) \in \pi(\mathcal{C})^c$ so $y^b \equiv [\pi^c(y)]^*1 \in \mathcal{C}$. For all $x \in \mathcal{C}$, we have

$$\omega(xy^b) = \langle y^b, x^* \rangle = \langle [\pi^c(y)]^*1, x^* \rangle = \langle 1, \pi^c(y)x^* \rangle$$
$$= \langle 1, \pi^*(x^*)y \rangle = \langle 1, x^*y \rangle = \langle x, y \rangle = \omega(y^*x).$$

Applying Lemma 7, ^{b} is an involution on $\mathcal{C}$. Conversely, suppose π is selfadjoint and there exists an involution ^{b} on $\mathcal{C}$ such that $\omega(y^*x) = \omega(xy^b)$ for all $x, y \in \mathcal{C}$. Then by (1), $\pi(\mathcal{C})^c$ is an algebra. If $C \in \pi(\mathcal{C})^c$, then for all $x, y \in \mathcal{C}$ we have

$$\langle Cy, x \rangle = \langle C1, y^*x \rangle = \langle (C1)x^b, y^* \rangle = \langle y, x(C1)^b \rangle.$$

Hence $\mathcal{C} \subseteq D(C^\dagger)$ and $C^\dagger | \mathcal{C} = \pi^c[(C1)^b] \in \pi^c(\mathcal{C}) = \pi(\mathcal{C})^c$. Thus $\pi(\mathcal{C})^c$ is a *-algebra.

(3) If $\pi(\mathcal{C})^c$ is a *-algebra, then π is selfadjoint so $\rho(x) = \pi^c(x)$ for all $x \in \mathcal{C}$. It follows from Theorem 6 that $\rho: \mathcal{C} \to \pi(\mathcal{C})^c$ is a weakly continuous,

linear bijection and it is clear that ρ is an anti-isomorphism. Finally, for all x, $y \in \mathcal{Q}$ we have

$$\rho(x^b)y = yx^b = y\left\{\left[\pi^c(x)\right]^*1\right\} = \left[\pi^c(x)\right]^*y = \left[\rho(x)\right]^*y. \quad \square$$

Notice that Theorem 8(3) gives a characterization of $\pi(\mathcal{Q})^c$ of the form $\pi(\mathcal{Q})^c = \rho(\mathcal{Q})$. We shall later discuss a similar characterization of $\pi(\mathcal{Q})'$.

5. The commutant. Let $(\mathcal{Q}, *, \omega)$ be a probability algebra and let $\mathcal{Q}_0 = \{y \in \mathcal{Q}: \mathcal{Q} \ni x \to \pi(x)y$ is bounded$\}$, $D(\pi^*)_0 = \{y \in D(\pi^*): \mathcal{Q} \ni x \to \pi^*(x)y$ is bounded$\}$.

Of course, $\mathcal{Q}_0 \subseteq D(\pi^*)_0$. If $y \in D(\pi^*)_0$, we define $\pi'(y)$ to be the unique bounded operator on H which satisfies $\pi'(y)x = \pi^*(x)y$ for all $y \in \mathcal{Q}$.

LEMMA 9. π' *is a weakly continuous, linear bijection from* $D(\pi^*)_0$ *onto* $\pi(\mathcal{Q})'$.

PROOF. This is similar to the proof of Theorem 6. $\square$

If π is selfadjoint we can, of course, replace $D(\pi^*)_0$ by $\mathcal{Q}_0$ in Lemma 9. This selfadjointness condition, which was also used in §4, is a very strong requirement which eliminates many cases of interest. For this reason, we shall impose the weaker, more reasonable, condition that π is *essentially selfadjoint*; that is, the closure $\bar{\pi}$ of π is selfadjoint. Furthermore, one of the difficulties with $\mathcal{Q}_0$ and $D(\pi^*)_0$ is that they might not be closed under the $*$ operation. This difficulty is circumvented by introducing a class of probability algebras (which we call closable) in which $*$ is a closable operator.

For $x, y \in \mathcal{Q}$ define the seminorms $\|x\|_y = \|yx\|$. The *induced topology* on $\mathcal{Q}$ is the topology generated by the collection of seminorms $\{\|x\|_y: y \in \mathcal{Q}\}$. This is the weakest topology on $\mathcal{Q}$ in which $\pi(x)$ is bounded for every $x \in \mathcal{Q}$. Let $\hat{\mathcal{Q}}$ be the completion of $\mathcal{Q}$ relative to the induced topology. Then $\mathcal{Q} \subseteq \hat{\mathcal{Q}} \subseteq H$ and $\hat{\mathcal{Q}}$ is a dense subspace of H. Since $\pi(x)$ is continuous on $\mathcal{Q}$ (in the induced topology), $\pi(x)$ has a unique continuous (in the induced topology) extension to $\hat{\mathcal{Q}}$ which we denote by $\bar{\pi}(x)$. We call $\bar{\pi}$ the *closure* of π. It can be shown that $\bar{\pi}$ is a $*$-representation of $\mathcal{Q}$ with domain $D(\bar{\pi}) = \hat{\mathcal{Q}}$, that $\bar{\pi} \subseteq \pi^*$, $(\bar{\pi})^* = \pi^*$, and $\bar{\pi}(\mathcal{Q})' = \pi(\mathcal{Q})'$ [13]. Thus π is essentially selfadjoint if and only if $\bar{\pi} = \pi^*$. In general, although $\pi(\mathcal{Q})'$ is a weakly closed, $*$-closed, linear manifold in $L(H)$, $\pi(\mathcal{Q})'$ need not be a von Neumann algebra [13]. If, however, π is essentially selfadjoint, then it is immediate that $\pi(\mathcal{Q})'$ is a von Neumann algebra.

LEMMA 10. (1) *The following statements are equivalent:*

(a) π *is essentially selfadjoint*;

(b) $C1 \in \hat{\mathcal{Q}}$ *for every* $C \in \pi(\mathcal{Q})^c$;

(c) *for any* $F < \omega$ *there exists a* $y \in \hat{\mathcal{Q}}$ *such that* $F(x) = \langle x, y \rangle$, *for all* $x \in \mathcal{Q}$.

(2) *If* $\pi(x)$ *is a bounded operator for every* $x \in \mathcal{Q}$, *then* π *is selfadjoint*.

PROOF. (1) Similar to Theorem 8(1). (2) If the $\pi(x)$ are bounded then $\hat{\mathcal{Q}} = H$ so $D(\bar{\pi}) = D(\pi^*) = H$. ☐

A probability algebra $(\mathcal{Q}, *, \omega)$ is *closable* if for any sequence $x_i \in \mathcal{Q}$ which satisfies $\lim \omega(x_i^* x_i) = 0$ and $\lim \omega(x_i x_i^*) = \lim_{i,j} \operatorname{Re} \omega(x_i x_j^*)$ we have $\lim \omega(x_i x_i^*) = 0$. A *lip algebra* is a *-algebra $(\mathcal{Q}, *)$ with an inner product $\langle \cdot, \cdot \rangle$ satisfying:

(5.1) $\langle 1, 1 \rangle = 1$, $\langle x^*x, 1 \rangle = \langle x, x \rangle$ for all $x \in \mathcal{Q}$;

(5.2) $x \mapsto x^*$ is closable in the completion H of $\mathcal{Q}$.

A lip algebra is more general than a Tomita left Hilbert algebra [17] in the sense that left multiplication is not assumed to be continuous; it is less general in the sense that an identity is assumed. The next lemma shows that there is essentially no difference between a closable probability algebra and a lip algebra.

LEMMA 11. (1) *Let $(\mathcal{Q}, *, \omega)$ be a closable probability algebra. If we define $\langle x, y \rangle = \omega(y^*x)$, then $(\mathcal{Q}, *, \langle \cdot, \cdot \rangle)$ is a lip algebra.*

(2) *Let $(\mathcal{Q}, *, \langle \cdot, \cdot \rangle)$ be a lip algebra. If we define $\omega(x) = \langle x, 1 \rangle$, then $(\mathcal{Q}, *, \omega)$ is a closable probability algebra such that $\langle x, y \rangle = \omega(y^*x)$.*

(3) *In any lip algebra, $\langle xy, z \rangle = \langle y, x^*z \rangle$.*

PROOF. (1) Clearly $\langle \cdot, \cdot \rangle$ is an inner product and (5.1) holds. To show $x \to x^*$ is closable, suppose $x_i \to 0$ and x_i^* is Cauchy. Then $\omega(x_i^* x_i) = \|x_i\|^2 \to 0$ and

$$\omega(x_i x_i^*) + \omega(x_j x_j^*) - 2\operatorname{Re} \omega(x_i x_j^*)$$
$$= \langle x_i^*, x_i^* \rangle + \langle x_j^*, x_j^* \rangle - 2\operatorname{Re}\langle x_j^*, x_i^* \rangle = \|x_i^* - x_j^*\|^2 \to 0.$$

Since x_i^* converges in H, $\omega(x_i x_i^*) = \|x_i^*\|^2$ converges. Hence $\operatorname{Re} \omega(x_i x_j^*)$ converges and $\lim_{i,j}\operatorname{Re} \omega(x_i x_j^*) = \lim \omega(x_i x_i^*)$. It follows that $\|x_i^*\|^2 = \omega(x_i x_i^*) \to 0$, so $x_i^* \to 0$.

(2) Since $\omega(x^*x) = \langle x^*x, 1 \rangle = \langle x, x \rangle = \|x\|^2$, we see that ω is a faithful state. The other condition for a closable probability algebra follows by reversing the steps in (1). That $\langle x, y \rangle = \omega(y^*x)$ follows from $\|x\|^2 = \omega(x^*x)$ and the polarization equation

$$\langle x, y \rangle = \tfrac{1}{2}\left[i\|x + iy\|^2 + \|x + y\|^2 - (1 + i)(\|x\|^2 + \|y\|^2) \right].$$

(3) This follows from (2). ☐

In the remainder of this section, $(\mathcal{Q}, *, \omega)$ will be a closable probability algebra. If S denotes the closure of $*$, then S is a closed conjugate-linear operator with $D(S) \supseteq \mathcal{Q}$. The adjoint S^* of S is a closed, densely defined, conjugate-linear operator and $\langle Sx, y \rangle = \langle S^*y, x \rangle$ for all $x \in D(S)$, $y \in D(S^*)$.

LEMMA 12. $D(S)$ *is invariant under S and $S^2x = x$ for every $x \in D(S)$.*

$D(S^*)$ *is invariant under* S^* *and* $S^{*2}x = x$ *for every* $x \in D(S^*)$.

PROOF. Let $x \in D(S), y \in D(S^*)$. Since S is the closure of $*$ and $D(*) = \mathcal{C}$, there exists a sequence $x_i \in \mathcal{C}$ such that $x_i \to x$ and $Sx_i \to Sx$. Hence

$$\langle S^*y, Sx \rangle = \lim\langle S^*y, Sx_i \rangle = \lim\langle S^*y, x_i^* \rangle$$
$$= \lim\langle Sx_i^*, y \rangle = \lim\langle x_i, y \rangle = \langle x, y \rangle.$$

Thus $Sx \in D(S)$ and $S^2x = x$. A similar argument holds for S^*. □

Let $\mathcal{C}' = \{y \in D(S^*) \cap D(\pi^*): \mathcal{C} \in x \to \pi^*(x)y$ is bounded$\}$. For $y \in \mathcal{C}'$, let $\pi'(y)$ be the unique bounded operator on H satisfying $\pi'(y)x = \pi^*(x)y$ for all $x \in \mathcal{C}$.

LEMMA 13. $y \in \mathcal{C}'$ *if and only if there exist a bounded operator* A *and a* $y_1 \in D(\pi^*)$ *such that* $Ax = \pi^*(x)y, A^*x = \pi^*(x)y_1$ *for every* $x \in \mathcal{C}$. *In this case* $y_1 = S^*y, \pi'(y) = A$.

PROOF. Suppose there exist an A and $y_1 \in D(\pi^*)$. For any $x \in \mathcal{C}$ we have

$$\langle x^*, y \rangle = \langle \pi(x^*)1, y \rangle = \langle 1, \pi^*(x)y \rangle$$
$$= \langle 1, Ax \rangle = \langle A^*1, x \rangle = \langle y_1, x \rangle.$$

Hence y is in the domain of the adjoint of $*$. But the adjoint of a closable operator equals the adjoint of its closure, so $y \in D(S^*)$ and $S^*y = y_1$. Also $x \to \pi(x)y$ is bounded on $\mathcal{C}$ and $\pi'(y) = A$. Conversely, suppose $y \in \mathcal{C}'$ and let $A = \pi'(y), y_1 = S^*y$. Then $Ax = \pi'(y)x = \pi^*(x)y$ and for every x, $z \in \mathcal{C}$ we have

$$\langle \pi(x^*)z, y_1 \rangle = \langle x^*z, S^*y \rangle = \langle y, z^*x \rangle = \langle y, \pi(z^*)x \rangle$$
$$= \langle \pi^*(z)y, x \rangle = \langle Az, x \rangle = \langle z, A^*x \rangle.$$

Hence $y_1 \in D(\pi^*)$ and $\pi^*(y_1) = A^*x$ for all $x \in \mathcal{C}$. □

THEOREM 14. *Let* $(\mathcal{C}, *, \omega)$ *be a closable probability algebra for which* π *is essentially selfadjoint. Then* $\mathcal{C}'$ *is a* $*$-*algebra under the product* $y_1 \circ y_2 = \pi'(y_1)y_2$ *and the involution* $y^* = S^*y$. *Furthermore,* π' *is a* $*$-*isomorphism of* $\mathcal{C}'$ *onto* $\pi(\mathcal{C})'$.

PROOF. That π' is an injective map from $\mathcal{C}'$ to $\pi(\mathcal{C})'$ is similar to the proof in Theorem 6. To show that π' is surjective, let $A \in \pi(\mathcal{C})'$. Then for all $x \in \mathcal{C}$ we have

$$\langle x^*, A1 \rangle = \langle A^*x^*, 1 \rangle = \langle A^*1, x \rangle.$$

Hence $A1 \in D(S^*)$ and $S^*(A1) = A^*1$. We conclude that $A1 \in \mathcal{C}'$. Since $Ax = \pi^*(x)A1, A^*x = \pi^*(x)A^*1$ for all $x \in \mathcal{C}$, it follows from Lemma 13 that $\pi'(A1) = A$ so π' is surjective. If $y \in \mathcal{C}'$, then by Lemma 13, $S^*y \in \mathcal{C}'$ and for all $x \in \mathcal{C}$ we have

$$\pi'(S^*y)x = \pi^*(x)S^*y = A^*x = \pi'(y)^*x. \tag{5.3}$$

Furthermore, if $x \in \mathcal{Q}$ and $y_1, y_2 \in \mathcal{Q}'$, then

$$\pi'(y_1)\pi'(y_2)x = \pi'(y_1)\pi^*(x)y_2 = \pi'(y_1)\bar{\pi}(x)y_2$$
$$= \pi^*(x)\pi'(y_1)y_2$$

and

$$\pi'(y_2)^*\pi'(y_1)^*x = \pi'(S^*y_2)\pi'(S^*y_1)x = \pi'(S^*y_2)\pi^*(x)S^*y_1$$
$$= \pi^*(x)\pi'(S^*y_2)S^*y_1.$$

By Lemma 13, $\pi'(y_1)y_2 \in \mathcal{Q}'$, $S^*\pi'(y_1)y_2 = \pi'(S^*y_2)S^*y_1$, and $\pi'[\pi'(y_1)y_2]$ $= \pi'(y_1)\pi'(y_2)$. It is now easy to verify that $\mathcal{Q}$ is an algebra under the product $y_1 \circ y_2 = \pi'(y_1)y_2$ and that π' is an isomorphism from $\mathcal{Q}'$ onto $\pi(\mathcal{Q})'$. The operation $y^* = S^*y$ is an involution since S^* is conjugate linear, $S^{*2} = I$, and

$$S^*(y_1 \circ y_2) = S^*\pi'(y_1)y_2 = \pi'(S^*y_2)S^*y_1 = y_2^* \circ y_1^*.$$

Finally, π' is a *-isomorphism by (5.3). $\quad\square$

COROLLARY. $\mathcal{Q}'$ *is a Tomita left Hilbert algebra.*

PROOF. $\mathcal{Q}'$ is a lip algebra, since for every $x \in \mathcal{Q}'$ we have

$$\langle x^* \circ x, 1 \rangle = \langle (S^*x) \circ x, 1 \rangle = \langle \pi'(S^*x)x, 1 \rangle$$
$$= \langle \pi'(x)^*x, 1 \rangle = \langle x, \pi'(x)1 \rangle = \langle x, x \rangle.$$

Furthermore, left multiplication is bounded. $\quad\square$

Our next result is analogous to a result of Tomita and Takesaki [17]. Our proof follows van Daele [3, p. 381] closely.

THEOREM 15. *Let* $(\mathcal{Q}, *, \omega)$ *be a closable probability algebra. Then there exist a bounded conjugate-linear operator* $J: H \to H$ *such that* $J^2 = I, J = J^*$ *and a positive selfadjoint (not necessarily bounded) operator* Δ *on* H *satisfying*:

(1) $\Delta = S^*S, \Delta^{-1} = SS^*$;

(2) $D(S) = D(\Delta^{1/2}), D(S^*) = D(\Delta^{-1/2})$;

(3) $J\Delta J = \Delta^{-1}$ *and* $Jf(\Delta)J = f^*(\Delta^{-1})$ *for every measurable function* f *on* $(0, \infty)$;

(4) $S = J\Delta^{1/2} = \Delta^{-1/2}J, S^* = J\Delta^{-1/2} = \Delta^{1/2}J$.

PROOF. Let H^* be the Hilbert space consisting of the set H with scalar multiplication $(\alpha, x) \to \alpha^*x$ and inner product $(x, y) \to \langle y, x \rangle$. Then S: $D(S) \to H^*$ is a closed linear operator with dense domain. Hence $\Delta = S^*S$ is a nonnegative selfadjoint operator on H. By the polar decomposition theorem [11, p. 282] there exists a partial isometry $J: H \to H^*$ such that $S = J\Delta^{1/2}$ and the initial domain of J is $\overline{R(S^*)}$. Now the range of S^* is dense since $x \in D(S^*)$ implies $x = S^*(S^*x)$, so $D(S^*) = R(S^*)$ (identifying H and H^*), and thus $\overline{R(S^*)} = H^*$. Hence J is unitary. Since $(S^*S)(SS^*) = I$ on $\mathcal{Q}$

we have $\Delta^{-1} = SS^*$ and Δ is positive selfadjoint. Since $S = S^{-1}$ we have $J\Delta^{1/2} = \Delta^{-1/2}J^*$ and $J\Delta^{1/2}J = \Delta^{-1/2}$. Also, $\Delta^{-1/2} = J^2J^*\Delta^{1/2}J$, and as J^2 is unitary and $J^*\Delta^{1/2}J$ is positive, it follows from the uniqueness of the polar decomposition of $\Delta^{-1/2}$ that $J^2 = I$ and, hence, $J = J^*$. (1), (2), and (4) now follow. To prove (3), notice that $J\Delta J = \Delta^{-1}$. If $g(\lambda) = \Sigma c_i \lambda^i$ is a polynomial, then

$$g(\Delta^{-1}) = \sum c_i (J\Delta J)^i = \sum c_i J\Delta^i J = J\left(\sum c_i^*\Delta^i\right)J = Jg^*(\Delta)J.$$

If f is a measurable function on $(0, \infty)$, it follows that $Jf(\Delta)J = f^*(\Delta^{-1})$. □

Notice that J satisfies the equations $\langle Jx, y \rangle = \langle Jy, x \rangle$ and $\langle Jx, Jy \rangle = \langle y, x \rangle$. If $\pi(x)$ is bounded, then $\mathcal{Q}'$ is dense in H [17]. In general, we have the following result.

COROLLARY. *Let $(\mathcal{Q}, *, \omega)$ be a closable probability algebra in which $\bar{\pi}$ is selfadjoint. If $\mathcal{Q}'$ is dense in H then $\pi(\mathcal{Q})'' = J\pi(\mathcal{Q})'J$.*

PROOF. By Tomita and Takesaki [17], we have $J\pi'(\mathcal{Q}')J = \pi'(\mathcal{Q}')'$. Hence
$$\pi(\mathcal{Q})'' = \pi'(\mathcal{Q}')' = J\pi'(\mathcal{Q}')J = J\pi(\mathcal{Q})'J. \square$$

6. Symmetric probability algebras. Let $(\mathcal{Q}, \#, \omega)$ be a probability algebra. If there exists a map $x \to x^b$ on $\mathcal{Q}$ such that $\omega(y^\# x) = \omega(xy^b)$ for every $x, y \in \mathcal{Q}$, then we call $(\mathcal{Q}, \#, \omega)$ a *symmetric* probability algebra. Theorem 8 gives a sufficient (but not necessary) condition for a probability algebra to be symmetric. A *rip algebra* is an algebra $\mathcal{Q}$ with an identity 1, an involution ^{b}, and an inner product $\langle \cdot, \cdot \rangle$ satisfying:

(6.1) $\langle xx^b, 1 \rangle = \langle x, x \rangle, \langle 1, 1 \rangle = 1$;

(6.2) $x \to x^b$ is closable on the completion H of $\mathcal{Q}$.

LEMMA 16. (1) *Let $(\mathcal{Q}, \#, \omega)$ be a symmetric probability algebra and let $\langle x, y \rangle = \omega(y^\# x)$. Then ^{b} is unique, $(\mathcal{Q}, \#, \langle \cdot, \cdot \rangle)$ is a lip algebra and $(\mathcal{Q}, ^b, \langle \cdot, \cdot \rangle)$ is a rip algebra. Furthermore, for every $x, y, z \in \mathcal{Q}$ we have*
$$\langle xy, z \rangle = \langle y, x^\# z \rangle = \langle x, zy^b \rangle.$$

(2) *Let $\mathcal{Q}$ be an algebra with an identity 1, two involutions $\#, ^b$, and an inner product $\langle \cdot, \cdot \rangle$ which satisfies $\langle x, x \rangle = \langle x^*x, 1 \rangle = \langle xx^b, 1 \rangle$. Define $\omega(x) = \langle x, 1 \rangle$. Then $(\mathcal{Q}, \#, \omega)$ is a symmetric probability algebra and $\langle x, y \rangle = \omega(y^\# x)$.*

(3) *A symmetric probability algebra is closable.*

(4) *A closable probability algebra $(\mathcal{Q}, *, \omega)$ is symmetric if and only if $\mathcal{Q} \subseteq D(S^*)$ and $S^*: \mathcal{Q} \to \mathcal{Q}$.*

PROOF. (1) Since $\langle x^\#, y \rangle = \langle y^b, x \rangle$, the domain of the adjoint of $^\#$ contains $\mathcal{Q}$. Hence this domain is dense and $^\#$ is closable. A similar argument shows that ^{b} is closable. The rest follows from Lemmas 7 and 11.

(2) Similar to the proof of Lemma 11.

(3) This follows from (1) and Lemma 11.

(4) If the closable probability algebra $(\mathcal{Q}, *, \omega)$ is symmetric, then as in the proof of (1), $\mathcal{Q} \subseteq D(S^*)$. Furthermore, for all $x, y \in \mathcal{Q}$ we have

$$\langle S^*x, y \rangle = \langle Sy, x \rangle = \langle y^*, x \rangle = \langle x^b, x \rangle.$$

Hence $S^*x = x \in \mathcal{Q}$ and $S^*: \mathcal{Q} \to \mathcal{Q}$. Conversely, if $\mathcal{Q} \subseteq D(S^*)$ and $S^*: \mathcal{Q} \to \mathcal{Q}$, define $x^b = S^*x$ for any $x \in \mathcal{Q}$. Then for all $x, y \in \mathcal{Q}$ we have

$$\omega(y^*x) = \langle x, y \rangle = \langle S^*y, Sx \rangle = \langle S^*y, x^* \rangle$$
$$= \omega[x(S^*y)] = \omega(xy^b). \quad \square$$

Let $(\mathcal{Q}, \#, \omega)$ be a symmetric probability algebra with inner product $\langle x, y \rangle = \omega(y^\#x) = \omega(xy^b)$. Let H be the Hilbert space completion of $\mathcal{Q}$. For x, $y \in \mathcal{Q}$ define the seminorms $\|x\|_y = \|yx\|$, $_y\|x\| = \|xy\|$. The *left (right) induced topology* on $\mathcal{Q}$ is the topology generated by the collection of seminorms $\{\| \ \|_y : y \in \mathcal{Q}\}$ ($\{_y\| \ \| : y \in \mathcal{Q}\}$). Let $\mathcal{Q}^l$ ($\mathcal{Q}^r$) be the completion of $\mathcal{Q}$ relative to the left (right) induced topology. Since $\mathcal{Q}$ is dense in $\mathcal{Q}^l$ ($\mathcal{Q}^r$) in the left (right) induced topology, for any $x \in \mathcal{Q}$ there exists a unique operator $\bar{\pi}(x)$ ($\bar{\rho}(x)$) on $\mathcal{Q}^l$ ($\mathcal{Q}^r$) such that $\bar{\pi}(x)y = xy$ ($\bar{\rho}(x)y = yx$) for all $y \in \mathcal{Q}$. Then $\bar{\pi}$ is a $\#$-representation of $\mathcal{Q}$ with domain $D(\bar{\pi}) = \mathcal{Q}^l \subseteq H$ and $\bar{\rho}$ is a b-antirepresentation of $\mathcal{Q}$ with domain $D(\bar{\rho}) = \mathcal{Q}^r \subseteq H$. Let S and F be the closures of $\#$ and b, respectively. Then $\mathcal{Q} \subseteq D(S), D(F)$.

LEMMA 17. *$D(S)$ is invariant under S and $S^2x = x$ for every $x \in D(S)$. $D(F)$ is invariant under F and $F^2x = x$ for every $x \in D(F)$. Furthermore $S = F^*$ and $F = S^*$.*

PROOF. The first part follows as in the proof of Lemma 12. For the last statement, let $x \in D(S)$ and $y \in D(F)$. There exist a sequence $x_i \in \mathcal{Q}$ such that $x_i \to x$, $Sx_i \to Sx$ and a sequence $y_i \in \mathcal{Q}$ such that $y_i \to y$, $Fy_i \to Fy$. We then obtain

$$\langle Sx, y \rangle = \lim \langle Sx_i, y_i \rangle = \lim \langle x_i^\#, y_i \rangle = \lim \langle y_i^b, x_i \rangle$$
$$= \lim \langle Fy_i, x_i \rangle = \langle Fy, x \rangle.$$

It follows that $S = F^*$ and $F = S^*$. $\quad \square$

LEMMA 18. *Let $A \in \rho(\mathcal{Q})'$, $B \in \pi(\mathcal{Q})'$.*
(1) $A*1 \in D(\pi^\#)$ if and only if $Ay \in D(S)$ for every $y \in \mathcal{Q}$ and in this case $S(Ay) = \pi^\#(y^\#)A*1$.
(2) $B*1 \in D(\rho^b)$ if and only if $By \in D(F)$ for every $y \in \mathcal{Q}$ and in this case $F(By) = \rho^b(y^b)B*1$.

PROOF. We shall prove (2), the proof of (1) being similar. Suppose $B*1 \in D(\rho^b)$. Then for every $x, y \in \mathcal{Q}$ we have

$$\langle x^\#, By \rangle = \langle B^*\pi(x^\#)1, y \rangle = \langle B*1, \pi(x)y \rangle$$
$$= \langle B*1, \rho(y)x \rangle = \langle \rho^b(y^b)B*1, x \rangle.$$

Hence $By \in D(F)$ and $F(By) = \rho^b(y^b)B^*1$. Conversely, suppose $By \in D(F)$ for every $y \in \mathcal{C}$. As in the proof of Theorem 14, $B^*1 \in D(F)$ and $F(B^*1) = B1$. Then for every $y, z \in \mathcal{C}$ we have

$$\langle \rho(z)y, B^*1 \rangle = \langle yz, B^*1 \rangle = \langle F(B^*1), z^{\#}y^{\#} \rangle = \langle B1, \pi(z^{\#})y^{\#} \rangle$$
$$= \langle B\pi(z)1, y^{\#} \rangle = \langle Bz, y^{\#} \rangle = \langle y, F(Bz) \rangle.$$

Hence $B^*1 \in D(\rho(z)^*)$ for every $z \in \mathcal{C}$ and so $B^*1 \in D(\rho^b)$. □

COROLLARY. *Let* $A \in \rho(\mathcal{C})'$, $B \in \pi(\mathcal{C})'$. *If* $\mathcal{C}^l = \mathcal{C}^r$ *and* π, ρ *are essentially selfadjoint, then* $By \in D(F)$, $Ay \in D(S)$ *and* $F(By) = \bar{\rho}(y^b)B^*1$, $S(Ay) = \bar{\pi}(y^{\#})A^*1$ *for every* $y \in \mathcal{C}$.

PROOF. If $\mathcal{C}^l = \mathcal{C}^r$ and π, ρ are essentially selfadjoint, then $D(\pi^{\#}) = \mathcal{C}^l = \mathcal{C}^r = D(\rho^b)$ and this subspace is invariant under $\rho(\mathcal{C})'$ and $\pi(\mathcal{C})'$. □

THEOREM 19. *Let* $(\mathcal{C}, {}^{\#}, \omega)$ *be a symmetric probability algebra for which* $\mathcal{C}^l = \mathcal{C}^r$ *and* π, ρ *are essentially selfadjoint. Then* $\rho(\mathcal{C})' \subseteq \pi(\mathcal{C})''$.

PROOF. Let $A \in \rho(\mathcal{C})'$, $B \in \pi(\mathcal{C})'$ and let $\mathcal{C} \ni x_i \to A1$. Applying the previous corollary, for any $x, y \in \mathcal{C}$ we have

$$\langle ABx, y \rangle = \langle Bx, A^*y \rangle = \langle S(A^*y), F(Bx) \rangle$$
$$= \langle \bar{\pi}(y^{\#})A1, \bar{\rho}(x^b)B^*1 \rangle = \langle A1, \bar{\pi}(y)\bar{\rho}(x^b)B^*1 \rangle$$
$$= \lim \langle x_i, \bar{\pi}(y)\bar{\rho}(x^b)B^*1 \rangle = \lim \langle y^{\#}x_i x, B^*1 \rangle$$
$$= \lim \langle \pi(y^{\#})\rho(x)x_i, B^*1 \rangle = \lim \langle x_i, \bar{\rho}(x^b)\bar{\pi}(y)B^*1 \rangle$$
$$= \langle A1, \bar{\rho}(x^b)\bar{\pi}(y)B^*1 \rangle = \langle \bar{\rho}(x)A1, \bar{\pi}(y)B^*1 \rangle$$
$$= \langle Ax, B^*y \rangle = \langle BAx, y \rangle.$$

Since A and B are bounded we have $AB = BA$ so $\rho(\mathcal{C})' \subseteq \pi(\mathcal{C})''$. □

It is natural to conjecture that $\rho(\mathcal{C})' = \pi(\mathcal{C})''$. In the example which follows, this is so even though $\mathcal{C}^l \neq \mathcal{C}^r$.

7. An example. Let $\mathcal{C}$ be the complex associative algebra with identity 1 generated by elements p and q satisfying the Heisenberg commutation relation $pq - qp = -i1$. We call $\mathcal{C}$ the *canonical algebra for one degree of freedom*. We denote by ${}^{\#}$ the unique involution of $\mathcal{C}$ under which the generators p and q are selfadjoint.

The *Schrödinger representation* π_0 of $(\mathcal{C}, {}^{\#})$ acts in the Hilbert space $H_0 = L^2(R)$, with domain the space $S_0 = \mathcal{S}(\mathbf{R})$ of infinitely differentiable rapidly decreasing functions, and is defined by $\pi_0(p) = p_0$, $\pi_0(q) = q_0$ where $p_0 = -id/dt$ and q_0 is multiplication by the coordinate $t \in \mathbf{R}$. The Schrödinger representation is faithful, selfadjoint and irreducible, meaning that its commutant $\pi(\mathcal{C})'$ consists of multiples of the identity [13]. The unit vector

$$f_0(t) = \pi^{-1/4}\exp(-t^2/2) \qquad (7.1)$$

is strongly cyclic [13].

We denote by $a^{\pm}$ the linear combinations, mutually adjoint under $^{\#}$,

$$a^+ = 2^{-1/2}(q - ip), \quad a^- = 2^{-1/2}(q + ip), \qquad (7.2)$$

satisfying

$$a^-a^+ - a^+a^- = 1,$$

and by $a_0^{\pm}$ their images under π_0. The vectors

$$f_n = \|(a_0^+)^n f\|^{-1}(a_0^+)^n f_0, \qquad n = 0, 1, \ldots,$$

form a maximal orthonormal set in H_0 contained in S_0. In fact f_n is the normalized Hermite function

$$f_n(t) = (\sqrt{\pi 2^n\, n!})^{-1/2}H_n(t)\exp(-t^2/2)$$

where

$$H_n(t) = (-1)^n\exp t^2\left(\frac{d}{dt}\right)^n\exp(-t^2)$$

is the nth Hermite polynomial. The action of $a_0^{\pm}$ on the f_n is

$$a_0^+ f_n = \sqrt{n+1}\, f_{n+1}, \quad a_0^- f_n = \begin{cases} 0 & \text{if } n = 0, \\ \sqrt{n}f_{n-1} & \text{if } n > 0. \end{cases} \qquad (7.3)$$

We denote by $\mathcal{Q}^{(m)}$ the subspace of $\mathcal{Q}$ comprising polynomials in p and q of degree $< m$. Using (7.3) it is not difficult to see that for $x \in \mathcal{Q}^{(m)}$,

$$\|\pi_0(x)f_n\| < C(x)n^{m/2}$$

where $C(x) > 0$ is independent of n. It follows that for each $\beta > 0$ the series

$$\omega_\beta(x) = (1 - e^{-\beta})^{-1}\sum_{n=0}^{\infty} e^{-n\beta}\langle\pi_0(x)f_n, f_n\rangle$$

converges for arbitrary $x \in \mathcal{Q} = \bigcup_{m=0}^{\infty}\mathcal{Q}^{(m)}$, and defines a faithful state ω_β of $(\mathcal{Q}, {}^{\#})$, called the *centered isotropic normal state* of parameter β (the terminology will be justified in §9).

LEMMA 19. $(\mathcal{Q}, {}^{\#}, \omega_\beta)$ *is a symmetric probability algebra.*

PROOF. Fix $x \in \mathcal{Q}$ and consider

$$\omega_\beta(a^+x) = (1 - e^{-\beta})^{-1}\sum_{n=0}^{\infty} e^{-\beta n}\langle a_0^+\pi_0(x)f_n, f_n\rangle$$

$$= (1 - e^{-\beta})^{-1}\sum_{n=0}^{\infty} e^{-\beta n}\langle\pi_0(x)f_n, a_0^- f_n\rangle$$

$$= (1 - e^{-\beta})^{-1}\sum_{n=0}^{\infty} e^{-\beta n}\sqrt{n}\,\langle\pi_0(x)f_n, f_{n-1}\rangle$$

together with

$$\omega_\beta(xa^+) = (1 - e^{-\beta})^{-1} \sum_{n=0}^{\infty} e^{-\beta n} \langle \pi_0(x)a_0^+ f_n, f_n \rangle$$

$$= (1 - e^{-\beta})^{-1} \sum_{n=0}^{\infty} e^{-\beta n} \sqrt{n+1} \, \langle \pi_0(x)f_{n+1}, f_n \rangle.$$

Comparing these expressions gives

$$\omega_\beta(a^+x) = e^{-\beta}\omega_\beta(xa^+),$$

and a similar argument gives

$$\omega_\beta(a^-x) = e^{\beta}\omega_\beta(xa^-).$$

Using (7.2), it follows that for arbitrary $x \in \mathcal{C}$,

$$\omega_\beta(px) = \omega_\beta(xp^\Delta), \quad \omega_\beta(qx) = \omega_\beta(xq^\Delta)$$

where

$$p^\Delta = \cosh \beta p - i \sinh \beta q, \quad q^\Delta = i \sinh \beta p + \cosh \beta q.$$

Since $\mathcal{C}$ is generated by p and q it follows that for arbitrary $x, y \in \mathcal{C}$,

$$\omega_\beta(y^\# x) = \omega_\beta(x\Delta(y^\#))$$

where Δ is the automorphism of $\mathcal{C}$ which maps the generators p and q to p^Δ and q^Δ, respectively. Setting $y^b = \Delta(y^\#)$, we have for all $x, y \in \mathcal{C}$,

$$\omega_\beta(y^\# x) = \omega_\beta(xy^b) \tag{7.4}$$

as required. $\square$

It is clear that the automorphism Δ is the restriction to $\mathcal{C}$ of the operator denoted also by Δ whose existence, together with that of the involutory isometric conjugate-linear operator J on the completion H of $\mathcal{C}$ in the inner product $\langle \, , \, \rangle$ determined by ω, is asserted in Theorem 15. However, in this case we have much more structure; in fact $(\mathcal{C}, \#, \omega)$ has the properties of a modular Hilbert algebra [17, Definition 1.1], with the exception of the continuity of left multiplication.

THEOREM 20. *There exists a complex one-parameter group $\{\Delta(z); z \in \mathbf{C}\}$ of automorphisms of $\mathcal{C}$ such that for all $x, y \in \mathcal{C}, z \in \mathbf{C}, t \in \mathbf{R}$:*
(a) $(\Delta(z)x)^\# = \Delta(-\bar{z})x^\#$;
(b) $\langle \Delta(z)x, y \rangle = \langle x, \Delta(\bar{z})y \rangle$;
(c) $\langle \Delta(1)x^\#, y^\# \rangle = \langle y, x \rangle$;
(d) $\langle \Delta(z)x, y \rangle$ *is an analytic function of $z \in \mathbf{C}$;*
(e) $\{x + \Delta(t)x : x \in \mathcal{C}\}$ *is dense in $\mathcal{C}$.*

PROOF. We set, for each $z \in \mathbf{C}$,

$$p^{\Delta(z)} = \cosh(\beta z)p - i\sinh(\beta z)q,$$

$$q^{\Delta(z)} = i\sinh(\beta z)p + \cosh(\beta z)q. \tag{7.5}$$

Since $p^{\Delta(z)}q^{\Delta(z)} - q^{\Delta(z)}p^{\Delta(z)} = -i1$, there is a unique automorphism $\Delta(z)$ of $\mathcal{C}$ under which $p^{\Delta(z)}$ and $q^{\Delta(z)}$ are the images of p and q. By comparing the images of p and q it is clear that $\Delta(z_1 + z_2) = \Delta(z_1)\Delta(z_2)$ for arbitrary z_1, $z_2 \in \mathbf{C}$, and that $\Delta(-z_1) = \Delta(z_1)^{-1}$. Thus $\{\Delta(z) : z \in \mathbf{C}\}$ is a complex one-parameter group. Since

$$p^{\Delta(z)\sharp} = \cosh(\beta\bar{z})p + i\sinh(\beta\bar{z})q = p^{\Delta(-\bar{z})} = p^{\sharp\Delta(-\bar{z})}$$

and, similarly, $q^{\Delta(z)\sharp} = q^{\sharp\Delta(-\bar{z})}$, the conjugate-anti-automorphisms $x \mapsto (\Delta(z)x)^{\sharp}$ and $x \mapsto \Delta(-z)x^{\sharp}$ agree on the generators p and q and are, hence, identical. This proves (a). To prove (b) note first that $\Delta(z)a^{+} = e^{-\beta z}a^{+}$, $\Delta(z)a^{-} = e^{\beta z}a^{-}$, whence $\Delta(z)(a^{+j}a^{-k}) = e^{-(j-k)\beta z}a^{+j}a^{-k}$ and so

$$\langle \Delta(z)(a^{+j}a^{-k}), a^{+l}a^{-m} \rangle = e^{-(j-k)\beta z}\langle a^{+j}a^{-k}, a^{+l}a^{-m} \rangle,$$

$$\langle a^{+j}a^{-k}, \Delta(\bar{z})(a^{+l}a^{-m}) \rangle = e^{-(l-m)\beta z}\langle a^{+j}a^{-k}, a^{+l}a^{-m} \rangle.$$

Now if $j - k \neq l - m$ we have

$$\langle a^{+j}a^{-k}, a^{+l}a^{-m} \rangle = \omega_{\beta}(a^{+m}a^{-l}a^{+j}a^{-k})$$

$$= (1 - e^{-\beta})^{-1} \sum_{n=0}^{\infty} e^{-\beta n}\langle a_0^{+m}a_0^{-l}a_0^{+j}a_0^{-k}f_n, f_n \rangle$$

$$= (1 - e^{-\beta})^{-1} \sum_{n=0}^{\infty} e^{-\beta n}\langle a_0^{+j}a_0^{-k}f_n, a_0^{+l}a_0^{-m}f_n \rangle$$

$$= 0,$$

as follows from (7.3) and the orthogonality of the f_n. It follows that for all values of j, k, l, m,

$$\langle \Delta(z)(a^{+j}a^{-k}), a^{+l}a^{-m} \rangle = \delta_{j-k,l-m}e^{-(j-k)\beta}\langle a^{+j}a^{-k}, a^{+l}a^{-m} \rangle$$

$$= \langle a^{+j}a^{-k}, \Delta(\bar{z})(a^{+l}a^{-m}) \rangle.$$

Since the elements $a^{+j}a^{-k}$, $j, k = 0, 1, \ldots,$ span $\mathcal{C}$, (b) is proved. (c) is an immediate consequence of (7.4) and the definition $x^{b} = \Delta x^{\sharp} = \Delta(1)x^{\sharp}$. (d) is clear from the fact that $\Delta(z)x$ is a polynomial, with coefficients in $\mathcal{C}$, in the analytic functions $\cosh z$, $\sinh z$. Finally, since $\Delta(t)(a^{+j}a^{-k}) = e^{-(j-k)\beta t}a^{+j}a^{-k}$ it is clear that the linear set $\{x + \Delta(t)x : x \in \mathcal{C}\}$ includes the set $\{a^{+j}a^{-k} : j, k = 0, 1, \ldots\}$, linear combinations of which span $\mathcal{C}$. Hence (e) holds. $\square$

The state ω_{β} satisfies the "KMS boundary condition" [6], [17] for the automorphisms $\{\Delta(it) : t \in \mathbf{R}\}$, meaning that for arbitrary $x, y \in \mathcal{C}$ there is a continuous function $F_{x,y}$ on the strip $0 \leqslant \text{Im } z \leqslant 1$, analytic in $0 < \text{Im } z < 1$, such that

$$F_{x,y}(t) = \omega_\beta\big((\Delta(it)x)y\big), \quad F_{x,y}(t+i) = \omega_\beta\big(y\Delta(it)x\big).$$

Indeed, if we define $F_{x,y}(z) = \omega_\beta((\Delta(iz)x)y)$, then $F_{x,y}$ is everywhere analytic and, hence, continuous, and, using (a), (b) and (c) of Theorem 20, we have

$$\begin{aligned}
F_{x,y}(t+i) &= \omega_\beta\big((\Delta(it-1)x)y\big) = \langle y, (\Delta(it-1)x)^{\#}\rangle\\
&= \langle y, \Delta(it+1)x^{\#}\rangle = \langle y, \Delta(it)\Delta(1)x^{\#}\rangle = \langle \Delta(-it)y, \Delta(1)x^{\#}\rangle\\
&= \overline{\langle \Delta(1)x^{\#}, \Delta(-it)y\rangle} = \overline{\langle(\Delta(-it)y)^{\#}, x\rangle} = \langle x, \Delta((-it)y)^{\#}\rangle\\
&= \langle x, \Delta(-it)y^{\#}\rangle = \langle \Delta(it)x, y^{\#}\rangle = \omega_\beta\big(y\Delta(it)x\big)
\end{aligned}$$

as required.

Motivated by the analogy with a modular Hilbert algebra, we introduce the conjugate-anti-automorphic map

$$*\colon \mathcal{C} \ni x \to x^* = \Delta\big(\tfrac{1}{2}\big)x^{\#}.$$

From (7.5) we obtain $p^{**} = p$, $q^{**} = q$ so $*$ is an involution. Since

$$\langle x^*, y^*\rangle = \langle \Delta\big(\tfrac{1}{2}\big)x^{\#}, \Delta\big(\tfrac{1}{2}\big)y^{\#}\rangle = \langle \Delta(1)x^{\#}, y^{\#}\rangle = \langle y, x\rangle,$$

by Theorem 20(d), $*$ is isometric and extends uniquely to a conjugate linear isometry J (in fact the operator so denoted of Theorem 15) from H to H. In this case we have

LEMMA 21. *J maps $\mathcal{C}^l$ continuously onto $\mathcal{C}^r$, and vice versa. For arbitrary $x \in \mathcal{C}$,*

$$\bar{\rho}(x) = J\bar{\pi}(x^*)J.$$

PROOF. By the isometry of $*$, for $x, y \in \mathcal{C}$,

$$_y\|x^*\| = \|x^*y\| = \|(y^*x)^*\| = \|y^*x\| = \|x\|_{y^*}. \tag{7.6}$$

To show that $J\colon \mathcal{C}^l \to \mathcal{C}^r$, let $x \in \mathcal{C}^l$ and let x_i be a net in $\mathcal{C}$ converging to x in the left induced topology. Then, in particular, $Jx_i \to Jx$ in the Hilbert space topology. Moreover, $\rho(y)Jx_i$ is Cauchy in H for every $y \in \mathcal{C}$, since

$$\|\rho(y)Jx_i - \rho(y)Jx_j\| = {}_y\|x_i^* - x_j^*\| = \|x_i - x_j\|_{y^*}$$

by (7.6). Hence $Jx \in D\overline{(\rho(y))}$, where $\overline{\rho(y)}$ is the closure of the operator $\rho(y)$. Since y was arbitrary, $Jx \in \bigcap_{y\in\mathcal{C}}D\overline{(\rho(y))} = D(\bar{\rho}) = \mathcal{C}^r$ as required. Extending $_y\|\ \|$ to $\mathcal{C}^r$ and $\|\ \|_y$ to $\mathcal{C}^l$, it follows from (7.5) that

$$_y\|Jx\| = \|x\|_{y^*}$$

for all $y \in \mathcal{C}$, $x \in \mathcal{C}^l$. Thus $J\colon \mathcal{C}^l \to \mathcal{C}^r$ is continuous in the induced topologies. A similar argument shows that J maps $\mathcal{C}^r$ continuously to $\mathcal{C}^l$ and the first statement of the lemma now follows. Finally for arbitrary $y \in \mathcal{C}^r = D(\bar{\rho})$, if y_i is a net in $\mathcal{C}$ converging in $\mathcal{C}^r$ to y we have for arbitrary $x \in \mathcal{C}$, since $\bar{\pi}(x^*)$ is continuous on $\mathcal{C}^l$,

$$J\bar{\pi}(x^*)Jy = \lim_i J\bar{\pi}(x^*)Jy_i = \lim_i J\bar{\pi}(x^*)y_i^* = \lim_i Jx^*y_i^*$$

$$= \lim_i J(y_ix)^* = \lim_i y_ix = \bar{\rho}(x)y. \quad \square$$

8. A commutation theorem. In this section we shall show that, if $(\mathscr{C}, {}^{\#}, \omega_\beta)$ is the probability algebra generated by the centered isotropic normal state ω_β of parameter β on the canonical algebra $(\mathscr{C}, {}^{\#})$, then the commutants $\pi(\mathscr{C})'$ and $\rho(\mathscr{C})'$ are commutants of each other, and that the isometry J induces a spatial anti-isomorphism from each to the other.

We recall [13] that $\pi(\mathscr{C})' = \bar{\pi}(\mathscr{C})'$ and $\rho(\mathscr{C})' = \bar{\rho}(\mathscr{C})'$. We shall construct a pair of representations π_1 and ρ_1 unitarily equivalent to the pair $\bar{\pi}$ and $\bar{\rho}$, whose commutants can be exhibited in explicit form.

We recall [14] that the domain $S_0 = \mathfrak{S}(\mathbf{R})$ of the Schrödinger representation π_0 is a nuclear space with its usual topology, which is in fact the topology on S_0 induced by π_0 [13], but is determined by the countable family of norms $\| \ \|_m$, $m = 0, 1, \ldots$, where

$$\|f\|_m^2 = \int \left| \left(-\frac{d^2}{dt^2} + t^2 + \frac{1}{2} \right)^m f(t) \right|^2 dt = \|\pi_0((a^-a^+)^m)f\|^2.$$

We denote by $H_0 \otimes \bar{H}_0$ the Hilbert space of Hilbert-Schmidt operators in $H_0 = L^2(\mathbf{R})$, and by $S_0 \otimes \bar{H}_0$ the subspace of $H_0 \otimes \bar{H}_0$ consisting of Hilbert-Schmidt operators whose ranges are contained in S_0. For $T \in S_0 \otimes \bar{H}_0$ and $x \in \mathscr{C}$, $\pi_0(x)T$ is clearly a well-defined operator from H_0 to S_0.

The authors are grateful to A. Katavolos for pointing out an error in an earlier version of the proof of the lemma which follows, and to S. L. Woronowicz for conversations which resulted in the present version.

LEMMA 22. *For arbitrary $T \in S_0 \otimes \bar{H}_0$ and $x \in \mathscr{C}$, $\pi_0(x)T \in S_0 \otimes \bar{H}_0$.*

PROOF. Since $p_0^2 + q_0^2$ is an essentially selfadjoint operator with spectrum $\{1, 2, 3, \ldots \}$, $(p_0^2 + q_0^2)^{-1}$ exists and is a Hilbert-Schmidt operator. We write

$$\pi_0(x)T = (p_0^2 + q_0^2)^{-1}(p_0^2 + q_0^2)\pi_0(x)T = (p_0^2 + q_0^2)^{-1}\pi_0(y)T$$

where $y = (p^2 + q^2)x$. Since the Hilbert-Schmidt operators form an ideal in the algebra of all bounded operators, it is sufficient to show that $\pi_0(y)T$ is bounded.

To do this we first show that T is closed as an operator from the Hilbert space H_0 to the topological space S_0. Indeed, if $g_n \to g$ in H_0 and $Tg_n \to f$ in S_0 then, in particular, $Tg_n \to f$ in the sense of the H_0-norm on S_0. Since T is Hilbert-Schmidt it is a closed operator from H_0 to H_0 and, hence, $Tg = f$.

Since S_0 is a complete metrizable space it follows from the closed graph theorem that T is continuous from H_0 to S_0. Since $\pi_0(y)$ is continuous from

S_0 to itself, $\pi_0(y)T$ is continuous from H_0 to S_0. But then $\pi_0(y)T$ is, in particular, bounded with respect to the H_0-norm on S_0 as required. □

It follows from Lemma 22 that, for $x \in \mathcal{C}$, $\pi_1(x)$: $T \to \pi_0(x)T$ is a well-defined operator in the Hilbert space $H_0 \otimes \overline{H}_0$, with invariant domain $S_0 \otimes \overline{H}_0$.

LEMMA 23. π_1 *is a selfadjoint* [#]*-representation of* $\mathcal{C}$.

PROOF. That π_1 is a representation of $\mathcal{C}$ follows at once from the fact that π_0 is a representation. Also, for arbitrary T_1, $T_2 \in S_0 \otimes H_0$, if $\{ g_j, j = 0, 1, \dots \}$ is a maximal orthonormal set in H_0, we have for arbitrary $x \in \mathcal{C}$,

$$\langle \pi_1(x)T_1, T_2 \rangle = \sum_{j=0}^{\infty} \langle (\pi_1(x)T_1)g_j, T_2 g_j \rangle$$

$$= \sum_{j=0}^{\infty} \langle \pi_0(x)T_1 g_j, T_2 g_j \rangle = \sum_{j=0}^{\infty} \langle T_1 g_j, \pi_0(x^{\#})T_2 g_j \rangle$$

$$= \sum_{j=0}^{\infty} \langle T_1 g_j, (\pi_1(x^{\#})T_2)g_j \rangle = \langle T_1, \pi_1(x^{\#})T_2 \rangle,$$

using the fact that π_0 is a [#]-representation.

To prove selfadjointness, let $Q \in D(\pi_1^*)$. Then for arbitrary $x \in \mathcal{C}$ the map $S_0 \otimes \overline{H}_0 \ni T \to \langle \pi_1(x)T, Q \rangle$ is a bounded linear functional in $S_0 \otimes \overline{H}_0$. Set $T = f \otimes \bar{g}$, with $f \in S_0$, $g \in H_0$ where, for $g_1, g_2 \in H_0$, $g_1 \otimes \bar{g}_2$ is the element $g \to \langle g, g_2 \rangle g_1$ of $H_0 \cap \overline{H}_0$. Then for some positive real number M we have

$$M\|f\| \|g\| = M\|f \otimes \bar{g}\| \geqslant |\langle \pi_1(x)f \otimes \bar{g}, Q \rangle|$$

$$= |\langle (\pi_0(x)f) \otimes \bar{g}, Q \rangle| = |\langle \pi_0(x)f, Qg \rangle|.$$

Hence the map $S_0 \ni f \to \langle \pi_0(x)f, Qg \rangle$ is a bounded linear functional in S_0 for all $x \in \mathcal{C}$. Hence $Qg \in D(\pi_0^*) = D(\pi_0) = S_0$ since π_0 is selfadjoint. Hence Q maps arbitrary $g \in H_0$ into S_0, that is $Q \in S_0 \otimes \overline{H}_0 = D(\pi_1)$. Hence π_1 is selfadjoint. □

LEMMA 24. *The commutant* $\pi_1(\mathcal{C})'$ *of* π_1 *is the set of bounded operators C in* $H_0 \otimes \overline{H}_0$ *of form $CT = TC_1$ for some bounded operator C_1 in H_0.*

PROOF. If T is a Hilbert-Schmidt and C_1 a bounded operator with operator bound $\|C_1\|_{\infty}$, then TC_1 is a Hilbert-Schmidt operator whose Hilbert-Schmidt norm satisfies $\|TC_1\| \leqslant \|T\| \|C_1\|_{\infty}$ [15]. Hence C: $T \to TC_1$ is a bounded operator in $H_0 \otimes \overline{H}_0$. Moreover, for arbitrary $Q, T \in S_0 \otimes \overline{H}_0$ and $x \in \mathcal{C}$ we have

$$\langle C\pi_1(x)Q, T \rangle = \langle \pi_1(x)QC_1, T \rangle = \langle QC_1, \pi_1(x)T \rangle = \langle CQ, \pi_1(x^{\#})T \rangle,$$

showing that $C \in \pi(\mathcal{C})'$.

Conversely, let $C \in \pi(\mathcal{Q})'$. For fixed $g, g' \in H_0$ define a sesquilinear form $\Phi_{g,g'}$ on $H_0 \times H_0$ by

$$\Phi_{g,g'}(h, h') = \langle Ch \otimes \bar{g}, h' \otimes \bar{g}' \rangle.$$

Since

$$|\Phi_{g,g'}(h, h')| \leqslant \|C\|_\infty \|h \otimes \bar{g}\| \, \|h' \otimes \bar{g}'\|$$
$$= \|C\|_\infty \|h\| \, \|g\| \, \|h'\| \, \|g'\|,$$

$\Phi_{g,g'}$ is bounded, hence of form

$$\Phi_{g,g'}(h, h') = \langle \Gamma(g, g')h, h' \rangle$$

for some bounded operator $\Gamma(g, g')$ with

$$\|\Gamma(g, g')\|_\infty \leqslant \|C\|_\infty \|g\| \, \|g'\|. \tag{8.1}$$

We prove that $\Gamma(g, g')$ is in the commutant of the Schrödinger representation and is thus a multiple of the identity. Thus, if $f, f' \in D(\pi_0) = S_0$, for arbitrary $x \in \mathcal{Q}$,

$$\langle \Gamma(g, g')\pi_0(x)f, f' \rangle = \Phi_{g,g'}(\pi_0(x)f, f')$$
$$= \langle C(\pi_0(x)f) \otimes \bar{g}, f' \otimes \bar{g}' \rangle = \langle C\pi_1(x)(f \otimes \bar{g}), f' \otimes \bar{g}' \rangle$$
$$= \langle Cf \otimes \bar{g}, \pi_1(x^\#)f' \otimes \bar{g}' \rangle = \langle Cf \otimes \bar{g}, (\pi_0(x^\#)f') \otimes \bar{g}' \rangle$$
$$= \Phi_{g,g'}(f, \pi_0(x^\#)f') = \langle \Gamma(g, g')f, \pi_0(x^\#)f' \rangle.$$

Hence we can write $\Gamma(g, g') = \gamma(g', g)I$ where γ is easily seen to be sesquilinear and, in view of (8.1), satisfies

$$|\gamma(g', g)| \leqslant \|C\|_\infty \|g\| \, \|g'\|.$$

Thus we can set $\gamma(g', g) = \langle C_1 g', g \rangle$ for some bounded operator C_1 on H_0. But then, for arbitrary $g, g', h, h' \in H_0$,

$$\langle Ch \otimes \bar{g}, h' \otimes \bar{g}' \rangle = \Phi_{g,g'}(h, h') = \langle \Gamma(g, g')h, h' \rangle$$
$$= \gamma(g, g')\langle h, h' \rangle = \langle h, h' \rangle \langle C_1 g', g \rangle = \langle h, h' \rangle \langle g', C_1^* g \rangle$$
$$= \langle h \otimes (\overline{C_1^* g}), h' \otimes \bar{g}' \rangle = \langle (h \otimes \bar{g})C_1, h' \otimes \bar{g}' \rangle.$$

Since elements of form $h \otimes \bar{g}$ are total in $H_0 \otimes \bar{H}_0$, it follows that $CT = TC_1$ for arbitrary $T \in H_0 \otimes \bar{H}_0$. $\square$

We equip the space $S_0 \otimes \bar{H}_0$ with the topology determined by the countable family of norms $\| \, \|_m$, $m = 0, 1, \ldots$, where $\|T\|_m = \|\pi_1((a^- a^+)^m)T\|$, and establish an analogue for the topological space $S_0 \otimes \bar{H}_0$ of the "N-representation" and the expansion in terms of Hermite functions f_n [14] for S_0. Thus we let $s(H_0)$ be the topological vector space consisting of sequences $\{g_n\}_{n=0}^\infty$ of vectors $g_n \in H_0$ such that, for $m = 0, 1, 2, \ldots$, $\sup_n n^m \|g_n\| < \infty$, in which the topology is that determined by the norms $\| \, \|_m$, $m = 0, 1, \ldots$, where

$$\|\{g_n\}_{n=0}^{\infty}\|_m^2 = \sum_n (n+1)^{2m}\|g_n\|^2.$$

LEMMA 25. *The map* $S_0 \otimes \bar{H}_0 \ni T \mapsto \{g_n = T^*f_n\}_{n=0}^{\infty}$ *is an isomorphism from the topological vector space* $S_0 \otimes \bar{H}_0$ *onto* $s(H_0)$ *whose inverse is given by* $s_0(H_0) \ni \{g_n\}_{n=0}^{\infty} \mapsto T$, *where*

$$T = \sum_{n=0}^{\infty} f_n \otimes \bar{g}_n,$$

the convergence being in the sense of the topology of $S_0 \otimes \bar{H}_0$.

PROOF. If $T \in S_0 \otimes \bar{H}_0$ and $g_n = T^*f_n$, then for $m = 0, 1, \ldots$,

$$\|T\|_m^2 = \|\pi_1((a^-a^+)^m)T\|^2 = \|(\pi_1((a^-a^+)^m)T)^*\|^2$$

$$= \sum_{n=0}^{\infty} \|(\pi_1((a^-a^+)^m)T)^*f_n\|^2.$$

Now since $f_n \in S_0$, for arbitrary $x \in \mathcal{C}$, $g \in H_0$,

$$\langle (\pi_1(x)T)^*f_n, g \rangle = \langle f_n, \pi_1(x)Tg \rangle = \langle f_n, \pi_0(x)(Tg) \rangle$$

$$= \langle \pi_0(x^\#)f_n, Tg \rangle = \langle T^*\pi_0(x^\#)f_n, g \rangle,$$

showing that $(\pi_1(x)T)^*f_n = T^*\pi_0(x^\#)f_n$. Hence, since $a_0^-a_0^+f_n = (n+1)f_n$ we obtain

$$\|T\|_m^2 = \sum_{n=0}^{\infty} \|T^*(a_0^-a_0^+)^m f_n\|^2 = \sum_{n=0}^{\infty} (n+1)^m\|T^*f_n\|^2.$$

Hence the map $T \to \{g_n = T^*f_n\}_{n=0}^{\infty}$ is isometric for each of the norms $\| \ \|_m$ in $S_0 \otimes \bar{H}_0$ and $s(H_0)$, respectively.

Now let $\{g_n\}_{n=0}^{\infty} \in s(H_0)$, and consider the partial sums $T_N = \sum_{n=0}^{N} f_n \otimes \bar{g}_n$, $N = 0, 1, \ldots$, each of which is in $S_0 \otimes \bar{H}_0$ since each $f_n \in S_0$. For $M > N$, by the orthonormality of the f_n, we have

$$\|T_N - T_M\|_m^2 = \left\|\sum_{n=N+1}^{M} f_n \otimes \bar{g}_n\right\|_m^2 = \left\|\pi_1((a^-a^+)^m)\sum_{n=N+1}^{M} f_n \otimes \bar{g}_n\right\|^2$$

$$= \left\|\sum_{n=N+1}^{M} (\pi_0((a^-a^+)^m)f_n) \otimes \bar{g}_n\right\|^2$$

$$= \left\|\sum_{n=N+1}^{M} (n+1)^m f_n \otimes \bar{g}_n\right\|^2$$

$$= \sum_{n=N+1}^{M} (n+1)^{2m}\|f_n \otimes \bar{g}_n\|^2 = \sum_{n=N+1}^{M} (n+1)^m\|g_n\|^2,$$

which converges to 0 as $M, N \to \infty$, since $\|\{g_n\}_{n=0}^{\infty}\|_m^2$ is finite. Hence

$\{T_N\}_{N=0}^\infty$ is Cauchy in $S_0 \otimes \overline{H}_0$ and so converges to some $T \in S_0 \otimes \overline{H}_0$. Moreover, since $T = \sum_{n=0}^\infty f_n \otimes \bar{g}_n$, in particular in the sense of convergence in Hilbert-Schmidt norm, we have $T^* = \sum_{n=0}^\infty g_n \otimes \bar{f}_n$ (Hilbert-Schmidt norm convergence), whence since the f_n are orthonormal, $T^* f_n = g_n$. Thus T is indeed the preimage of $\{g_n\}_{n=0}^\infty$.

Thus we have an invertible linear map between the two spaces which is isometric for each of the norms defining the respective topologies. Hence the two spaces are isomorphic as claimed. $\square$

Since for each $\beta > 0$ the sequence $\{g_n = e^{-n\beta/2} f_n\}_{n=0}^\infty$ is in $s_0(H_0)$, the operator

$$\Omega_\beta = (1 - e^{-\beta})^{-1/2} \sum_{n=0}^\infty e^{-n\beta/2} f_n \otimes \bar{f}_n$$

is a well-defined element of $S_0 \otimes \overline{H}_0$.

LEMMA 26. *For arbitrary* $x \in \mathcal{C}$,

$$\omega_\beta(x) = \langle \pi_1(x)\Omega_\beta, \Omega_\beta \rangle.$$

PROOF.

$$\langle \pi_1(x)\Omega_\beta \Omega_\beta \rangle = \sum_{n=0}^\infty \langle (\pi_1(x)\Omega_\beta) f_n, \Omega_\beta f_n \rangle$$

$$= \sum_{n=0}^\infty \langle \pi_0(x)\Omega_\beta f_n, \Omega_\beta f_n \rangle$$

$$= (1 - e^{-\beta})^{-1} \sum_{n=0}^\infty e^{-n\beta} \langle \pi_0(x) f_n, f_n \rangle$$

$$= \omega_\beta(x). \quad \square$$

We recall [13] that a vector $\Omega \in S_0 \otimes \overline{H}_0$ is *strongly cyclic* for the representation π_1 if for arbitrary $T \in S_0 \otimes \overline{H}_0$ there exists a sequence x_n of elements of $\mathcal{C}$ such that $\pi_1(x_n)\Omega$ converges to T in the topology determined by the seminorms $\| \ \|_y, y \in \mathcal{C}$, where $\|T\|_y = \|\pi_1(y)T\|$. Since the powers of $a^- a^+$ dominate $\mathcal{C}$ [13] this topology coincides with the one already introduced determined by the norms $\| \ \|_m, m = 0, 1, \ldots$.

LEMMA 27. Ω_β *is strongly cyclic for* π_1.

PROOF. We first compute the kernel of the Hilbert-Schmidt operator Ω_β, considered as an integral operator in $H_0 = L^2(\mathbf{R})$. Using Mehler's formula for Hermite polynomials,

$$\sum_{n=0}^\infty (n!)^{-1} \left(\tfrac{1}{2} z\right)^n H_n(s) H_n(t)$$

$$= (1 - z^2)^{-1/2} \exp\left\{(1 - z^2)^{-1}(2zst - z^2(s^2 + t^2))\right\},$$

together with the defining formulae for Ω_β and f_n, this is

$$F_\beta(s, t) = (1 - e^{-\beta})^{-1/2} \sum_{n=0}^{\infty} e^{-n\beta/2} f_n(s) f_n(t)$$

$$= \pi^{-1/2} \exp\left\{ -\tfrac{1}{2} \coth\left(\tfrac{1}{2} \beta \right)(s^2 + t^2) + \operatorname{cosech}\left(\tfrac{1}{2} \beta \right) st \right\}.$$

By considering $\langle \pi_1(x)\Omega_\beta g, f \rangle = \langle \Omega_\beta g, \pi_0(x^*) f \rangle$ for $f \in S_0$ it is easily seen that for $x = \sum_{m,n} c_{mn} p^m q^n \in \mathcal{C}$, the kernel of $\pi_1(x)\Omega_\beta$ is given by

$$F_x(s, t) = \sum c_{mn} \left(-i \frac{\partial}{\partial s} \right)^m s^n F_\beta(s, t) = P_x(s, t) F_\beta(s, t)$$

where P_x is a complex polynomial in two indeterminates. Conversely, every such polynomial is of form P_x for some $x \in \mathcal{C}$. To see this, observe that P_1 is the identity polynomial, that $P_{qx}(s, t) = sP_x(s, t)$, and that, since the kernel of $\pi_1(px)\Omega_\beta$ is

$$\left(-i \frac{\partial}{\partial s} \right) P_x(s, t) F_\beta(s, t)$$

$$= \left(i \left(\coth\left(\tfrac{1}{2} \beta \right) s - \operatorname{cosech}\left(\tfrac{1}{2} \beta \right) t \right) P_x(s, t) - i \frac{\partial}{\partial s} P_x(s, t) \right) F_\beta(s, t),$$

$$P_{px}(s, t) = -i \operatorname{cosech}\left(\tfrac{1}{2} \beta \right) t P_x(s, t) + i \coth\left(\tfrac{1}{2} \beta \right) P_{qx}(s, t) + P'(s, t)$$

where P' has degree less than that of P. It is now clear by induction on the degree of P that an arbitrary polynomial P is of form P_x for some $x \in \mathcal{C}$.

Since the quadratic form in the exponential in $F_\beta(s, t)$ is nonsingular, the class of functions PF_β is dense in the Schwarz space $S(R^2)$. Thus, given arbitrary $F \in S(R^2)$, there exists a sequence $\{x_N\}_{N=0}^{\infty}$ of elements $x_N \in \mathcal{C}$ such that the kernels $P_{x_N} F_\beta$ converge to F in the Schwarz topology of $S(R^2)$. In particular, $P_{x_N} F_\beta$ converges to F in each of the norms $\| \ \|_m$ on $S(R^2)$, where

$$\|F\|_m^2 = \int_{R^2} \left| \left(-\frac{\partial^2}{\partial s^2} + t^2 + \frac{1}{2} \right) F(s, t) \right|^2 ds \, dt.$$

Now each $F \in S(R^2)$ is the kernel of a Hilbert-Schmidt operator T_F in $H_0 \otimes \overline{H}_0$. Let us show that $T_F \in S_0 \otimes \overline{H}_0$ and that $\|F\|_m = \|T_F\|_m$. To do this, observe that if

$$F(s, t) = \sum_{n,n'} c_{n,n'} f_n(s) f_{n'}(t)$$

is the two-dimensional Hermite expansion of F then

$$\|F\|_m^2 = \sum_{n,n'} (n + 1)^{2m} |c_{n,n'}|^2,$$

while T_F is given by

$$T_F = \sum_{n,n'} c_{n,n'} f_n \otimes \bar{f}_{n'}$$

where the convergence is in Hilbert-Schmidt norm, and this can be expressed as

$$T_F = \sum_n f_n \otimes \bar{g}_n$$

where $g_n = \sum_{n'} \bar{c}_{n,n'} f_{n'}$ and, in consequence,

$$\|\{g_n\}_{n=0}^{\infty}\|_m^2 = \sum_n (n+1)^{2m} \left\| \sum_{n'} \bar{c}_{nn'} f_{n'} \right\|^2$$

$$= \sum_{n,n'} (n+1)^{2m} |c_{n,n'}|^2 = \|F\|_m^2 < \infty.$$

It follows from Lemma 25 that $T_F \in S_0 \otimes \bar{H}_0$ with

$$\|T_F\|_m^2 = \|\{g_n\}_{n=0}^{\infty}\|_m^2 = \|F\|_m^2.$$

Now let $T \in S_0 \otimes \bar{H}_0$ be arbitrary; the strong cyclicity of Ω_β will be established if we can find elements x_N, $N = 0, 1, \ldots$, of $\mathcal{C}$ such that for $m = 0, 1, 2, \ldots$, $\|T - \pi_1(x_N)\Omega_\beta\|_m \to 0$ as $N \to \infty$. We write the "Hermite expansion" of T as

$$T = \sum_{n=0}^{\infty} f_n \otimes \bar{g}_n, \qquad g_n = T^* f_n \in H_0,$$

and recall that this converges in the topology of $S_0 \otimes \bar{H}_0$, so that

$$\left\| T - \sum_{n=0}^{N} f_n \otimes \bar{q}_n \right\|_m \to 0$$

as $n \to \infty$ for all m. Now S_0 is dense in the Hilbert space H_0, hence there exist elements $f_{nN} \in S_0$ such that $\|g_n - f_{nN}\| < e^{-N}$. We set $T_N = \sum_{n=0}^{N} f_n \otimes \bar{f}_{nN}$. Then

$$\|T - T_N\|_m \leq \left\| T - \sum_{n=0}^{N} f_n \otimes \bar{g}_n \right\|_m + \left\| \sum_{n=0}^{N} f_n \otimes (\bar{g}_n - f_{nN}) \right\|_m$$

$$= \left\| T - \sum_{n=0}^{N} f_n \otimes \bar{g}_n \right\|_m + \left\{ \sum_{n=0}^{N} (n+1)^{2m} e^{-2N} \right\}^{1/2}.$$

Hence $\|T - T_N\|_m \to 0$ as $N \to \infty$, for all $m = 0, 1, \ldots$. Now the kernel of T_N is $F_N(s, t) = \sum_{n=0}^{N} f_n(s) \bar{f}_{nN}(t)$, which is clearly an element of $\mathcal{S}(R^2)$. It follows that there exist x_{NM}, $M = 0, 1, \ldots$, such that for all $m = 0, 1, \ldots$, $\|F_N - P_{X_{NM}} F_\beta\|_m \to 0$, or, what is the same thing, $\|T_N - \pi_1(x_{NM})\Omega_\beta\|_m \to 0$, as $M \to \infty$.

Now choose a sequence $\{M_0(N)\}_{N=0}^{\infty}$ of nonnegative numbers such that, as $N \to \infty$, $\|T_N - \pi_1(x_{N,M_0(N)})\Omega_\beta\|_0 \to 0$. Assuming inductively that a sequence $\{M_m(N)\}_{N=0}^{\infty}$ of nonnegative numbers has been found such that, as $N \to \infty$,

$\|T_N - \pi_1(x_{N,M_m(N)})\Omega_\beta\|_{m'} \to 0$ for all $m' \leqslant m$, choose a subsequence $\{M_{m+1}(N)\}_{N=0}^\infty$ of $\{M_m(N)\}_{N=0}^\infty$ such that $\|T_N - \pi_1(x_{N,M_{m+1}(N)})\Omega_\beta\|_{m+1} \to 0$. Finally let x_N be defined as $x_N = x_{N,M_N(N)}$. Then $\|T_N - \pi_1(x_N)\Omega_\beta\|_m \to 0$ as $N \to \infty$ for all $m = 0, 1, \ldots$. Hence by the triangle inequality $\|T - \pi(x_N)\Omega_\beta\|_m \to 0$, $m = 0, 1, \ldots$, as was to be proved. $\square$

Lemmas 26 and 27 show that $(H_0 \otimes \overline{H}_0, \pi_1, \Omega_\beta)$ is a Gel'fand-Naimark-Segal triple for the state ω_β of $(\mathcal{C}, {}^\#)$ in the sense of [13]. But $(H, \overline{\pi}, 1)$ is trivially a second such triple. By the uniqueness of the Gel'fand-Naimark-Segal construction, there exists a unitary transformation W_β from H onto $H_0 \otimes \overline{H}_0$ under which $\overline{\pi}$ is unitarily equivalent to π_1, and such that $W_\beta\overline{\pi}(x)1 = \pi_1(x)\Omega_\beta$ for all $x \in \mathcal{C}$. In particular, it follows from Lemma 24 that $\overline{\pi}$ is selfadjoint. We define the ^{b}-antirepresentation ρ_1 by $\rho_1(x) = W_\beta\overline{\rho}(x)W_\beta^{-1}$ and the conjugate-linear isometry J_1 from $H_0 \otimes \overline{H}_0$ to itself by $J_1 = W_\beta J W_\beta^{-1}$. From Lemma 27 it is clear that J_1 maps the domain of π_1 bijectively to that of ρ_1 and vice versa, and that for arbitrary $x \in \mathcal{C}$, $\rho_1(x) = J_1\pi_1(x^*)J_1$.

LEMMA 28. *For arbitrary* $T \in H_0 \otimes \overline{H}_0$, $J_1T = T^*$, *the operator adjoint of* T.

PROOF. Since $T \mapsto T^*$ and J_1 are both conjugate linear isometries in $H_0 \otimes \overline{H}_0$, we may assume without loss of generality that T belongs to the dense subspace $W_\beta(\mathcal{C})$, say

$$T = W_\beta x = \pi_1(x)\Omega_\beta.$$

Then $J_1T = W_\beta Jx = W_\beta x^* = \pi_1(x^*)\Omega_\beta$. Thus we must show that $(\pi_1(x)\Omega_\beta)^* = \pi_1(x^*)\Omega_\beta$. We may further assume without loss of generality that x is of form $a^{-m}a^{+n}$.

For arbitrary $f \in S_0$, since a_0^+ is continuous in the topology of S_0,

$$a_0^+\Omega_\beta f = (1 - e^{-\beta})^{-1/2}\sum_{n=0}^\infty e^{-n\beta/2}\langle f, f_n\rangle a_0^+ f_n$$

$$= (1 - e^{-\beta})^{-1/2}\sum_{N=0}^\infty \sqrt{n+1}\, e^{-n\beta/2}\langle f, f_n\rangle f_{n+1}$$

$$= (1 - e^{-\beta})^{-1/2}e^{\beta/2}\sum_{n=0}^\infty e^{-(n+1)\beta/2}\langle f, a_0^- f_{n+1}\rangle f_{n+1}$$

$$= (1 - e^{-\beta})^{-1/2}e^{\beta/2}\sum_{n'=1}^\infty e^{-n'\beta/2}\langle f, a_0^- f_{n'}\rangle f_{n'}$$

$$= (1 - e^{-\beta})^{-1/2}e^{\beta/2}\sum_{n'=0}^\infty e^{-n'\beta/2}\langle f, a_0^- f_{n'}\rangle f_{n'}$$

$$= e^{\beta/2}\Omega_\beta a_0^+ f.$$

Since $a_0^+ f \in S_0$ we have by repetition

$$a_0^{+n} \Omega_\beta f = e^{n\beta/2} \Omega_\beta a_0^{+n} f,$$

and, by a similar argument,

$$a_0^{-m} a_0^{+n} \Omega_\beta f = e^{-(m-n)\beta/2} \Omega_\beta a_0^{-m} a_0^{+n} f.$$

Hence, for arbitrary $g \in H_0$,

$$\langle a_0^{-m} a_0^{+n} \Omega_\beta f, g \rangle = e^{-(m-n)\beta/2} \langle \Omega_\beta a_0^{-m} a_0^{+n} f, g \rangle$$

$$= e^{-(m-n)\beta/2} \langle a_0^{-m} a_0^{+n} f, \Omega_\beta g \rangle$$

$$= e^{-(m-n)\beta/2} \langle f, a_0^{-n} a_0^{+m} \Omega_\beta g \rangle.$$

Since S_0 is dense in H_0, it follows that

$$\left(\pi_1(x)\Omega_\beta\right)^* = \left(a_0^{-m} a_0^{+n} \Omega_\beta\right)^* = e^{-\beta(m-n)/2} a_0^{-n} a_0^{+m} \Omega_\beta.$$

But it follows easily from (7.5) and the definition of $*$ that

$$x^* = (a^{-m} a^{+n})^* = e^{-\beta(m-n)/2} a^{-n} a^{+m}.$$

Hence $(\pi_1(x)\Omega_\beta)^* = \pi_1(x^*)\Omega_\beta$ as required. $\square$

Now we can prove the main result of this section.

THEOREM 29. *The commutants $\pi(\mathcal{Q})'$ and $\rho(\mathcal{Q})'$ of π and ρ are commutants of each other as von Neumann algebras. Moreover, the operator J induces a spatial anti-isomorphism from either one of $\pi(\mathcal{Q})'$, $\rho(\mathcal{Q})'$ to the other.*

PROOF. It is sufficient to prove that the commutants of π_1 and ρ_1 are commutants of each other, spatially isomorphic under J_1. Since $\rho_1(x) = J_1 \pi_1(x^*) J_1$, it is clear that $\rho_1(\mathcal{Q})' = J_1 \pi_1(\mathcal{Q})' J_1$. By Lemma 24 $\rho_1(\mathcal{Q})'$ thus consists of all operators of form $T \to (T^* C_1)^* = C_1^* T_1$ where C_1 is a bounded operator on H_0. Thus $\rho_1(\mathcal{Q})'$ comprises all premultiplications by bounded operators in H_0, whereas $\pi_1(\mathcal{Q})'$ consists of all such post-multiplications. But it is well known [4] that these von Neumann algebras are each other's commutants. $\square$

9. A uniqueness theorem and its consequences. We recall that, for fixed $\beta > 0$, the involution ^{b} on the canonical algebra $\mathcal{Q}$ defined by $y^b = \Delta(y^\#)$, where Δ is the automorphism of $\mathcal{Q}$ under which the images of p and q are $p^\Delta = \cosh \beta p - i \sinh \beta q$ and $q^\Delta = i \sinh \beta p + \cosh \beta q$, makes $(\mathcal{Q}, ^\#, \omega_\beta)$ a symmetric probability algebra. The following theorem shows that ω_β is determined uniquely by ^{b}.

THEOREM 30. *Let ω be a state of $(\mathcal{Q}, ^\#)$ such that for all $x, y \in \mathcal{Q}$, $\omega(y^\# x) = \omega(xy^b)$. Then $\omega = \omega_\beta$.*

PROOF. Since $\Delta x = x^{\#b}$ we have, for arbitrary $x, y \in \mathcal{Q}$,

$$\omega(xy) = \omega(y\Delta x). \tag{9.1}$$

Using the facts that

$$\Delta a^+ = e^{-\beta}a^+, \quad \Delta a^- = e^{\beta}a^- \tag{9.2}$$

and

$$a^-a^+ - a^+a^- = 1, \tag{9.3}$$

we deduce that, for $r = 1, 2, \ldots$,

$$\omega((a^-a^+)^r) = e^{\beta}\omega((a^+a^-)^r) = e^{\beta}\omega((a^-a^+ - 1)^r)$$

$$= e^{\beta}\sum_{s=0}^{r} \binom{r}{s}(-1)^s\omega((a^-a^+)^s). \tag{9.4}$$

Equations (9.4) constitute a recurrence relation which determines the quantities $\omega((a^-a^+)^r)$, $r = 1, 2, \ldots$, once $\omega((a^-a^+)^0) = 1$ is fixed. Since our argument is applicable equally well to ω_β it follows that $\omega((a^-a^+)^r) = \omega_\beta((a^-a^+)^r)$ for all $r = 0, 1, 2, \ldots$. Since $a^{-r}a^{+r}$ can be expressed using (9.3) as a linear combination of $1, a^-a^+, (a^-a^+)^2, \ldots, (a^-a^+)^r$ it follows in turn that, for $r = 0, 1, 2 \ldots$,

$$\omega(a^{-r}a^{+r}) = \omega_\beta(a^{-r}a^{+r}). \tag{9.5}$$

On the other hand when $r \neq s$ we have, using (9.1) and (9.2),

$$\omega(a^{-r}a^{+s}) = e^{r\beta}\omega(a^{+s}a^{-r}) = e^{(r-s)\beta}\omega(a^{-r}a^{+s})$$

so that $\omega(a^{-r}a^{+s}) = 0$. Since by the same argument $\omega_\beta(a^{-r}a^{+s}) = 0$ we deduce that, for $r, s = 0, 1, \ldots$ and $r \neq s$,

$$\omega(a^{-r}a^{+s}) = \omega_\beta(a^{-r}a^{+s}). \tag{9.6}$$

The theorem follows from (9.5) and (9.6) together with the fact that $\mathcal{C}$ is generated by a^+ and a^-. $\square$

As a first application of Theorem 30 we show that ω_β is the unique state satisfying the KMS boundary conditions for the automorphisms $\{\Delta(it):$ $t \in R\}$ of Theorem 20. Thus, suppose ω is a KMS state, so that for arbitrary $x, y \in \mathcal{C}$ there is a continuous function $F_{x,y}$ on the strip $0 \leqslant \text{Im } z \leqslant 1$, analytic on $0 < \text{Im } z < 1$ such that

$$F_{x,y}(t) = \omega((\Delta(it)x)y), \quad F_{x,y}(t + i) = \omega(y\Delta(it)x).$$

Define the function $G_{x,y}$ on C by $G_{x,y}(z) = \omega((\Delta(iz)x)y)$. Then $G_{x,y}$ is everywhere analytic and coincides with $F_{x,y}$ on the real axis. Hence $G_{x,y}$ coincides with $F_{x,y}$ everywhere on the domain of definition of the latter, that is, we have

$$F_{x,y}(z) = \omega((\Delta(iz)x)y), \quad 0 \leqslant \text{Im } z \leqslant 1.$$

Now we can write, for arbitrary $x, y \in \mathcal{C}$,

$$\omega(y^{\#}x) = \omega\big((\Delta(-1)\Delta(1)y^{\#})x\big) = F_{\Delta(1)y^{\#},x}(i)$$

$$= \omega\big(x\Delta(0)\Delta(1)y^{\#}\big) = \omega(xy^{b}).$$

It follows from Theorem 30 that $\omega = \omega_{\beta}$.

The involutions $^{\#}$ and ^{b} and the modular automorphisms $\Delta(z)$, $z \in \mathbf{C}$, are *graded*, meaning that they leave invariant each subspace $\mathscr{C}^{(m)}$ of elements of degree $\leqslant m$. The next lemma shows that, modulo a graded automorphism ν of $\mathscr{C}$ (which is determined by its action on the generators p and q, of form

$$\binom{p}{q} \to \binom{p^{\nu}}{q^{\nu}} = A\binom{p}{q} + \begin{pmatrix} c_1 & I \\ c_2 & I \end{pmatrix},$$

where A is a 2×2 complex unimodular matrix and c_1, c_2 are complex constants) every graded probability algebra structure on $\mathscr{C}$ is equivalent to one of the algebras $(\mathscr{C}, ^{\#}, \omega_{\beta})$, $0 < \beta < \infty$.

LEMMA 31. *Let ω be a linear form on $\mathscr{C}$, and suppose that there exist graded involutions † and $^{\natural}$ of $\mathscr{C}$ such that $(\mathscr{C}, ^{\dagger}, \omega)$ is a symmetric probability algebra for which $\omega(y^{\dagger}x) = \omega(xy^{\natural})$ for all x and $y \in \mathscr{C}$. Then there exist a positive number β and a graded automorphism ν of $\mathscr{C}$ such that for all $x \in \mathscr{C}$, $\omega(x) = \omega_{\beta}(\nu(x))$, $x^{\dagger} = \nu^{-1}((\nu x)^{\#})$ and $x^{\natural} = \nu^{-1}((\nu x)^{b})$.*

PROOF. Let D be the automorphism $x \to x^{\dagger\natural}$ of $\mathscr{C}$. Then for arbitrary x, $y \in \mathscr{C}$ we have

$$\omega(xy) = \omega(yDx), \tag{9.7}$$

and D leaves each $\mathscr{C}^{(m)}$ invariant. Moreover, D is symmetric and positive with respect to the inner product $\langle x, y \rangle = \omega(y^{\dagger}x)$, being the restriction to $\mathscr{C}$ of the positive selfadjoint modular operator of Theorem 15 for the closable probability algebra $(\mathscr{C}, ^{\dagger}, \omega)$. Hence the restriction to $\mathscr{C}^{(1)}$ of D is fully reducible. If it is the identity then setting $x = p$ and $y = q$ in (9.7) gives $\omega(pq) = \omega(qp)$, which is impossible since $\omega(pq) - \omega(qp) = \omega(pq - qp) = -i$. Hence D possesses an eigenvector a, say, in $\mathscr{C}^{(1)}$ for which the corresponding eigenvalue $\lambda \neq 1$. From (9.7) we have $\omega(ax) = \lambda\omega(xa)$ for arbitrary $x \in \mathscr{C}$, whence, since ω is †-positive, $\omega(xa^{\dagger}) = \omega(ax^{\dagger}) = \lambda\omega(x^{\dagger}a) = \bar{\lambda}\omega(a^{\dagger}x)$. Since λ is real (D being symmetric) and nonzero (D being an automorphism) and since ω is faithful, it follows that $Da^{\dagger} = \lambda^{-1}a^{\dagger}$. Now the identity 1 is an eigenvector of D in $\mathscr{C}^{(1)}$ whose eigenvalue is 1. Thus the elements 1, a, $a^{\dagger}$ of $\mathscr{C}^{(1)}$, being eigenvectors belonging to distinct eigenvalues, are linearly independent. Writing

$$a = cp + dq + e1, \quad a^{\dagger} = c'p + d'q + e'1,$$

we must therefore have $cd' - c'd \neq 0$, and hence $aa^{\dagger} - a^{\dagger}a$ is a nonzero multiple of the identity. Since $aa^{\dagger}$, $a^{\dagger}a$ are †-selfadjoint this multiple is real and, by appropriate normalisation of a, can be taken to be either $+1$ or -1.

The latter case reduces to the former if we exchange the roles of a and $a^\dagger$, hence we may assume that

$$aa^\dagger - a^\dagger a = 1. \qquad (9.8)$$

Since the pair $(a, a^\dagger)$ and (a^+, a^-) are generators of $\mathcal{C}$ satisfying the same commutation relation there is a graded automorphism ν of $\mathcal{C}$ for which $\nu a = a^-$, $\nu a^\dagger = a^+$. Since ν maps the mutually †-adjoint generators $(a, a^\dagger)$ to the mutually $^\#$-adjoint generators (a^-, a^+), the involution † is given by $x^\dagger = \nu^{-1}((\nu x)^\#)$.

From (9.8) we have

$$\omega(aa^\dagger) - \omega(a^\dagger a) = 1,$$

while from (9.7)

$$\omega(aa^\dagger) = \lambda\omega(a^\dagger a).$$

From these equations it follows that $\lambda > 1$; we write $\lambda = e^\beta$ for β positive. Then $Da = e^\beta a$, $Da^\dagger = e^{-\beta}a^\dagger$. Hence $\nu D\nu^{-1}a^- = e^\beta a^-$, $\nu D\nu^{-1}a^+ = e^{-\beta}a^+$. Comparison with (9.2) shows that the automorphisms $\nu D\nu^{-1}$ and Δ coincide on the generators $a^\pm$ of $\mathcal{C}$, hence they are identical. Since $x^\natural = D(x^\dagger)$ we deduce that the involution $^\natural$ is given by

$$x^\natural = \nu^{-1}\Delta\nu x^\dagger = \nu^{-1}\Delta(\nu x)^\# = \nu^{-1}(\nu x)^b.$$

The lemma now follows by applying Theorem 30 to the state ω' of $(\mathcal{C}, ^\#)$, where $\omega'(x) = \omega(\nu^{-1}(x))$, which is $^\#$-positive, since for all $x \in \mathcal{C}$,

$$\omega'(x^\# x) = \omega\left(\nu^{-1}\left[\nu\left((\nu^{-1}x)^\dagger\right)x\right]\right) = \omega\left((\nu^{-1}x)^\dagger \nu^{-1}x\right) > 0,$$

and which satisfies

$$\omega'(y^\# x) = \omega\left(\nu^{-1}\left[\nu\left((\nu^{-1}y)^\dagger\right)x\right]\right) = \omega\left((\nu^{-1}y)^\dagger \nu^{-1}x\right)$$

$$= \omega\left(\nu^{-1}x(\nu^{-1}y)^\natural\right) = \omega\left(\nu^{-1}\left[y\nu((\nu^{-1}x))\right]^\natural\right) = \omega'(xy^b). \quad \square$$

As a final application of Theorem 30 we shall show that the states ω_β are, in a certain sense, indeed analogues of normal distribution in classical probability, as suggested by the terminology used to describe them. (Our argument may be compared with that of [2].)

We denote by $\mathcal{C}_n$ the algebraic tensor product $\mathcal{C} \otimes \cdots \otimes \mathcal{C}$ of n copies of $\mathcal{C}$, which we equip with the product involution denoted also by $^\#$ for which $(x_1 \otimes \cdots \otimes x_n)^\# = x_1^\# \otimes \cdots \otimes x_n^\#$. We set

$$p_j = 1 \otimes \cdots \otimes 1 \otimes \overset{(j)}{p} \otimes 1 \otimes \cdots \otimes 1,$$

$$q_j = 1 \otimes \cdots \otimes 1 \otimes \overset{(j)}{q} \otimes 1 \otimes \cdots \otimes 1$$

and observe that the elements

$$p^{(n)} = n^{-1/2}(p_1 + \cdots + p_n), \quad q^{(n)} = n^{-1/2}(q_1 + \cdots + q_n)$$

are $^{\#}$-selfadjoint and satisfy the Heisenberg relation $p^{(n)}q^{(n)} - q^{(n)}p^{(n)} = -i1$. It follows that there is a $^{\#}$-isomorphism ι_n from $\mathcal{C}$ into $\mathcal{C}^{(n)}$ under which p and q map to $p^{(n)}$ and $q^{(n)}$.

Now let ω be an arbitrary state of $(\mathcal{C}, ^{\#})$. The product state $\omega \otimes \cdots \otimes \omega$ on $(\mathcal{C}_n, ^{\#})$ is characterized by the property $\omega \otimes \cdots \otimes \omega(x_1 \otimes \cdots \otimes x_n) = \omega(x_1) \cdots \omega(x_n)$ where $x_1, \ldots, x_n \in \mathcal{C}$, and the formula

$$\omega^{(n)}(x) = \omega \otimes \cdots \otimes \omega(\iota_n(x))$$

defines a new state $\omega^{(n)}$ of $(\mathcal{C}, ^{\#})$. $\omega^{(n)}$ is the canonical analogue of the integral resulting from the rescaled n-fold convolution $\mu^{(n)}$ of a probability measure μ on R, defined by

$$\int_R f(t)\,\mu^{(n)}(dt) = \int_{R^n} f\big(n^{-1/2}(t_1 + \cdots + t_n)\big)\,\mu(dt_1) \cdots \mu(dt_n).$$

We recall [1] that the distribution determined by μ is said to be stable if, for each $n = 1, 2, \ldots,$ there exist real numbers c_n and d_n with $c_n > 0$ such that

$$\int f(t)\,\mu^{(n)}(dt) = \int f(c_n t + d_n)\,\mu(dt)$$

(this is not the usual definition, but is readily seen to be equivalent to it), that stable distributions are of significance insofar as they are possible limit distributions of sums of identically distributed random variables, and that stable distributions for which the variance is finite are normal. Thus motivated, we shall say that a state ω of $(\mathcal{C}, ^{\#})$ is *stable* if for each $n = 1, 2, \ldots$ there is a graded automorphism ν_n of $(\mathcal{C}, ^{\#})$ such that $\omega^{(n)}(x) = \omega(\nu_n(x))$ for all $x \in \mathcal{C}$.

We denote by ω_∞ the state of $(\mathcal{C}, ^{\#})$ given by $\omega_\infty(x) = \langle \pi_0(x) f_0, f_0 \rangle$.

Before stating our final theorem we require

LEMMA 32. *Let ω be an arbitrary state of $(\mathcal{C}, ^{\#})$. Then there exist a graded automorphism ν of $(\mathcal{C}, ^{\#})$ and a possibly infinite positive number β such that for all elements $x \in \mathcal{C}^{(2)}$ of degree not exceeding 2, $\omega(x) = \omega_\beta(\nu(x))$.*

PROOF. By replacing ω by ω', where $\omega'(x) = \omega(\nu_1(x))$ and ν_1 is the graded automorphism of $(\mathcal{C}, ^{\#})$ which maps p to $p - \omega(p)1$ and q to $q - \omega(q)1$, we may assume that $\omega(p) = \omega(q) = 0$. We consider the *covariance matrix*

$$\Gamma(\omega) = \begin{bmatrix} \omega(p^2) & \omega\big(\tfrac{1}{2}(pq + qp)\big) \\ \omega\big(\tfrac{1}{2}(pq + qp)\big) & \omega(q^2) \end{bmatrix},$$

which is the matrix of the quadratic form

$$\mathbf{R}^2 \ni (s, t) \to \omega\big((sp + tq)^2\big).$$

$\Gamma(\omega)$ is thus positive-definite, and by the Heisenberg uncertainty inequality we have also $\det \Gamma(\omega) \geqslant \frac{1}{4}$. It follows that there is a real unimodular 2×2 matrix A and a real number $\sigma^2 \geqslant 1$ such that

$$\Gamma(\omega) = \tfrac{1}{2}\sigma^2 A\,{}^tA.$$

Noting that for finite β, $\Gamma(\omega_\beta) = \frac{1}{2} \coth \beta I$, while $\Gamma(\omega_\infty) = \frac{1}{2} I$, where I is the 2×2 identity matrix, it follows that we can write

$$\Gamma(\omega) = A\,{}^t\Gamma(\omega_\beta)A$$

for some β, $0 < \beta \leqslant \infty$. But then if ν is the graded automorphism of $(\mathcal{Q}, {}^{\#})$ for which the images p'' and q'' of p and q are given by

$$\begin{pmatrix} p'' \\ q'' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix},$$

we have $\omega(x) = \omega_\beta(\nu(x))$ whenever x is p^2, $\frac{1}{2}(pq + qp)$ or q^2, and hence, since $\omega(x) = \omega_\beta(\nu(x))$ for all $x \in \mathcal{Q}^{(1)}$, the lemma is proved. $\square$

THEOREM 33. *A state ω of $(\mathcal{Q}, {}^{\#})$ is stable if and only if it is of the form $\omega(x) = \omega_\beta(\nu(x))$, where ν is a graded automorphism of $(\mathcal{Q}, {}^{\#})$ and β is a possibly infinite positive number.*

PROOF. We prove first that states of the form described are stable. Since it is clear from the definition that if ω is stable so is ω', where $\omega'(x) = \omega(\nu(x))$ for some graded automorphism ν, it suffices to show that ω_β is stable for all β, $0 < \beta \leqslant \infty$. In fact we shall prove that $\omega_\beta^{(n)} = \omega_\beta$ for $n = 1, 2, \ldots$.

In the case of finite β, using the fact that ι_n is both a ${}^{\#}$ and a ^{b}-isomorphism (where the product involution ^{b} on $\mathcal{Q}_n$ satisfies $(x_1 \otimes \cdots \otimes x_n)^b = x_1^b \otimes \cdots \otimes x_n^b$ for all $x_1, \ldots, x_n \in \mathcal{Q}$), we have, for arbitrary $x, y \in \mathcal{Q}$, writing

$$\iota_n(x) = \sum_j x_1^j \otimes \cdots \otimes x_n^j, \quad \iota_n(y) = \sum_k y_1^k \otimes \cdots \otimes y_n^k$$

with $x_i^j, y_l^k \in \mathcal{Q}$,

$$\omega_\beta^{(n)}(y^{\#}x) = \omega_\beta \otimes \cdots \otimes \omega_\beta\left(\sum_{j,k} y_1^{k\#}x_1^j \otimes \cdots \otimes y_n^{k\#}x_n^j\right)$$

$$= \sum_{j,k} \omega_\beta\big(y_1^{k\#}x_1^j\big) \cdots \omega_\beta\big(y_n^{k\#}x_n^j\big)$$

$$= \sum_{j,k} \omega_\beta\big(x_1^j y_1^{kb}\big) \cdots \omega_\beta\big(x_n^j y_n^{kb}\big)$$

$$= \omega_\beta \otimes \cdots \otimes \omega_\beta\left(\sum_{j,k} x_1^j y_1^{kb} \otimes \cdots \otimes x_n^j y_n^{kb}\right)$$

$$= \omega_\beta^{(n)}(xy^b).$$

It follows from Theorem 30 that $\omega_\beta^{(n)} = \omega_\beta$ as asserted.

In the case $\beta = \infty$ we note that ω_∞ is characterized by the fact that $\omega_\infty(a^{+r}a^{-s}) = \delta_{r,0}\delta_{s,0}$ and that, in consequence, writing

$$a_j^\pm = 1 \otimes \cdots \otimes 1 \otimes \overset{(j)}{a^\pm} \otimes 1 \otimes \cdots \otimes 1,$$

$$\omega_\infty^{(n)}(a^{+r}a^{-s})$$

$$= \omega_\infty \otimes \cdots \otimes \omega_\infty \left(n^{-(r+s)/2}(a_1^+ + \cdots + a_n^+)^r (a_1^- + \cdots + a_n^-)^s \right)$$

$$= \delta_{r,0}\delta_{s,0},$$

as may be seen from the multinomial expansion of $(a_1^+ + \cdots + a_n^+)^r$ and $(a_1^- + \cdots + a_n^-)^s$. Hence $\omega_\infty^{(n)} = \omega_\infty$.

Conversely let ω be a stable state of $(\mathcal{C}, {}^\#)$. By Lemma 32 there exist β, $0 < \beta \leqslant \infty$, and a graded ${}^\#$-automorphism ν of $\mathcal{C}$ such that $\omega(x) = \omega_\beta(\nu(x))$ for all $x \in \mathcal{C}^{(2)}$. Since if ω is stable, so too is ω', where $\omega'(x) = \omega(\nu^{-1}(x))$, we may assume that in fact ω coincides with ω_β on $\mathcal{C}^{(2)}$. We shall prove that under this assumption $\omega = \omega_\beta$.

Suppose first that $\beta = \infty$. Then $\omega(a^+a^-) = \omega_\infty(a^+a^-) = 0$. It follows from Schwarz's inequality that ω vanishes on the left ideal in $\mathcal{C}$ generated by a^- and on the right ideal generated by a^+. Hence $\omega(a^{+r}a^{-s}) = \delta_{r,0}\delta_{s,0} = \omega_\infty(a^{+r}a^{-s})$ for all $r, s = 0, 1, 2 \ldots$. Hence $\omega = \omega_\infty$.

Now let β be finite. For arbitrary $x, y \in \mathcal{C}^{(1)}$,

$$\omega(y^\# x) = \omega_\beta(y^\# x) = \omega_\beta(xy^b) = \omega(xy^b).$$

We shall prove that $\omega(y^\# x) = \omega(xy^b)$ for arbitrary $x, y \in \mathcal{C}$; from this it will follow from Theorem 30 that $\omega = \omega_\beta$. It is sufficient to consider the case when $y = r$ and x is of form $r^1 \cdots r^N$ where each of $r, r^1, \ldots, r^N$ is either p or q. The proof is by induction on N, the case $N = 1$ being already established.

Since ω is stable, for $n = 1, 2, \ldots$ there is a graded automorphism ν_n of $(\mathcal{C}, {}^\#)$ such that $\omega^{(n)}(z) = \omega(\nu_n(z))$ for arbitrary $z \in \mathcal{C}$. Since $\omega^{(n)}(p) = n^{1/2}\omega(p) = 0$ and $\omega^{(n)}(q) = n^{-1/2}\omega(q) = 0$, the automorphism ν_n acts on p and q by

$$\begin{pmatrix} p^{\nu_n} \\ q^{\nu_n} \end{pmatrix} = A_n \begin{pmatrix} p \\ q \end{pmatrix}$$

where A_n is a real unimodular 2×2 matrix, and the covariance matrices $\Gamma(\omega)$ and $\Gamma(\omega^{(n)})$ of ω and $\omega^{(n)}$ are related by

$$\Gamma(\omega^{(n)}) = A_n'\Gamma(\omega)A_n. \tag{9.9}$$

But it is easily verified from the definition of $\omega^{(n)}$, using the vanishing of $\omega(p)$ and $\omega(q)$, that $\Gamma(\omega^{(n)})$ and $\Gamma(\omega)$ are in fact equal. Since $\Gamma(\omega) = \Gamma(\omega_\beta)$ and the latter is a multiple of the identity, it follows from (9.9) that A_n is an orthogonal matrix and hence, being proper, is a rotation matrix. Hence we

can write

$$p^{\nu_n} = \cos\theta_n p + \sin\theta_n q, \quad q^{\nu_n} = -\sin\theta_n p + \cos\theta_n q$$

for some real θ_n, and also

$$p^{\nu_n^{-1}} = \cos\theta_n p - \sin\theta_n q, \quad q^{\nu_n^{-1}} = \sin\theta_n p + \cos\theta_n q. \tag{9.10}$$

Now consider

$$\omega(y^{\#}x) - \omega(xy^b) = \omega(y^{\#}x - xy^b) = \omega^{(n)}\big(\nu_n^{-1}(y^{\#}x - xy^b)\big)$$

$$= \omega^{(n)}\big(y'^{\#}x' - x'y'^b\big)$$

where, for $z \in \mathcal{C}$, $z' = \nu_n^{-1}z$. Writing $r'_j = 1 \otimes \cdots \otimes 1 \otimes \overset{(j)}{r'} \otimes 1 \otimes \cdots \otimes$

1, $r_j^{(k)'} = 1 \otimes \cdots \otimes 1 \otimes \overset{(j)}{r}\,^{(k)'} \otimes 1 \otimes \cdots \otimes 1$, and using the definition of $\omega^{(n)}$ this becomes

$$n^{-(N+1)/2}\omega \otimes \cdots$$

$$\otimes \overset{(n)}{\omega} \left[\left(\sum_{j=1}^{n} r'_j \right)^{\#} \prod_{k=1}^{N} \left(\sum_{j=1}^{n} r_j^{(k)'} \right) - \prod_{k=1}^{N} \left(\sum_{j=1}^{n} r_j^{(k)'} \right) \left(\sum_{j=1}^{n} r'_j \right)^{b} \right].$$

Expressing the two products of sums as sums of products we observe that corresponding products which involve more than one value of the suffix j cancel by the inductive hypothesis. The terms remaining are

$$n^{-(N+1)/2} \sum_{j=1}^{n} \omega \otimes \cdots \otimes \overset{(n)}{\omega} \left(r_j'^{\#} \prod_{k=1}^{N} r_j^{(k)'} - \prod_{k=1}^{N} (r_j^{(k)'}) r_j'^{b} \right)$$

$$= n^{-(N-1)/2}\omega \left(r'^{\#} \prod_{k=1}^{N} r'^{(k)} - \prod_{k=1}^{N} r'^{(k)} r'^{b} \right)$$

$$= n^{-(N-1)/2}\omega\big(\nu_n^{-1}(y^{\#}x - xy^b)\big).$$

Now from (9.10) it is clear that $\omega(\nu_n^{-1}(y^{\#}x - xy^b))$ is a polynomial in $\cos\theta_n$ and $\sin\theta_n$ whose coefficients do not depend on n. Hence $\omega(\nu_n^{-1}(y^{\#}x - xy^b))$ remains bounded as $n \to \infty$. Taking this limit we deduce that $\omega(y^{\#}x) - \omega(xy^b) = 0$ as required. $\square$

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